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VOL. 74 FEBRUARY, 1948 No. 2

TECHNICAL PAPERS

AND

DISCUSSIONS

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Discussion in Mar., June, Sept., 1947, Feb., 1948		Closed*
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* Publication of closing discussion pending.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

SHEARING STRESS DISTRIBUTION IN BOX GIRDERS WITH MULTIPLE WEBS

BY A. S. NEIMAN,[1] ASSOC. M. ASCE

SYNOPSIS

A method for finding the distribution of shearing stress in box girders with multiple webs is presented in this paper. In effect, the structure is allowed to distort under an assumed set of shearing forces. A correction is then introduced to satisfy conditions of geometry known to exist. The particular example solved indicates that in some parts of the structure the shearing stresses are considerably greater than the average.

INTRODUCTION

In computing the shear stress in a box girder with multiple webs, the formula frequently used is

$$v = \frac{V Q}{I t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which v is the unit shear stress at any point; V is the external shearing force; Q is the statical moment of the sectional area above the plane considered, about the neutral axis; I is the moment of inertia of the area of the cross section; and t is the combined thickness of members cut at the section. Eq. 1 gives the shear stress for a beam with single web of thickness t, or for a symmetrical box girder section with two webs of combined thickness t. For a girder with more than two web plates, such as is used frequently in heavy grillage foundations, or for a ship with longitudinal bulkheads, Eq. 1 is inadequate because it gives an average shear-stress value for all web members at the section. The actual shear stress may vary considerably from the average value. This is particularly true where web members are not of equal thickness, as is frequently the case for ships with longitudinal bulkheads.

The distribution of shear stress throughout a multiple-web beam, such as a ship with longitudinal bulkheads or a vessel with bulkheads that do not extend the full height, can be obtained by the mechanics of "shear flow" and the ap-

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by July 1, 1948.

[1] Asst. Chf. Structural Engr., Vacuum Concrete, Inc., Philadelphia, Pa.

plication of the principles of continuity to contiguous cells. The term "shear flow" as used herein denotes shear force per unit of linear measurement. Therefore, the shear flow at a point is the shear stress multiplied by the thickness of the metal at the point. The concept of shear flow is used extensively in the design of airplane structures.[2] The cross section of a vessel under external shearing forces is subjected to shearing stresses which may be considered as "flowing" through the section. This shear flow must be in equilibrium with the external shear force at the section, or with a combination of the external shear and the shear resulting from twisting moments. Moreover, the shear distribution must be such that contiguous cells undergo the same angular distortion. Therefore, the solution of the problem of shear distribution in a complex structure such as a ship with three or more shear-carrying webs, or with multiple decks, requires the application of the principles of geometry and statics. Two cases of shear distribution will be considered.

Notation

The following letter symbols adopted for use in this paper, and in its discussions, conform essentially with American Standard Letter Symbols for Structural Analysis, prepared by a committee of American Standards Association with Society participation:

A = area of one or more cells in the cross section of a tanker; ΔA = an increment of area of a web, or member = $t\,\Delta L$;
F = "function of"; F(I) and F(II) are simplifying substitution quantities defined by the left-hand side of Eqs. 4*a* and 4*b*, respectively;
G = modulus of elasticity in shear;
I = moment of inertia of the area of the cross section;
L = length of a web, or member; ΔL = an increment of length;
M = bending moment; M_t = external twisting moment;
Q = statical moment of the sectional area above the plane considered, about the neutral axis;
q = shear flow, in pounds (or kips) per unit of length:

q_a and q_b = shear flows introduced into cells I and II;
q_{BC} = shear flow at point B in member BC;
q_{GH} = shear flow at point G in member GH;
q_0 = known shear at any point;
q_1 = assumed shear flow;
q'_1 = average assumed shear flow;

t = thickness of a web or combined thickness of the members cut at a section;
V = external shearing force;
v = unit shear stress at any point;
z = moment arm; distance from the neutral axis to the centroid of the increment of area (ΔA) bounded by the plane upon which the shear is known and the plane upon which the value of shear is desired; and
θ = angle of twist.

[2] "Airplane Design," by K. D. Wood, Purdue Univ., Lafayette, Ind., 1941.

Case I. No Torsional Moment at Section

If no twisting moments exist at the section, the shearing stress must be the result of increments of longitudinal flexure only. The procedure for determining the distribution of shear stress for this case can be described in four steps:

1. Assume a distribution of shear flow q_1 consistent with the formulas:

$$q_1 = \frac{V}{I} \Sigma z \, \Delta A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

or

$$q_1 = q_0 \pm \frac{V}{I} z \, \Delta A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

in which V is the total shear at the section; I is the total moment of inertia about the neutral axis; q_0 is a known shear at any point; and $z \, \Delta A$ is the statical moment, about the neutral axis of the increment of area bounded by the plane upon which shear is known and the plane upon which the value of shear is desired.

2. Compute

$$2 A_1 G \theta = \Sigma q_1 \frac{\Delta L}{t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which A_1 is the area of one cell; G is the modulus of elasticity in shear; and $\frac{\Delta L}{t}$ is the ratio of increment of length to corresponding thickness. Eq. 3 gives the angle of twist, θ, under the assumed shear flow, q_1, in each cell.

3. Since in this example no torsion is acting on the structure, the angle of twist for each cell equals zero. Hence a shear flow acting around each cell can be introduced so that the resultant angle of twist of the cell is zero. This procedure can be followed for each cell by the use of equations similar to Eq. 3. For example, for a two cell structure (Fig. 1), the shear flows q_a and q_b are introduced into cells I and II, respectively, as corrections to the assumed shear flow q_1 so that the resultant angle of twist will be zero.

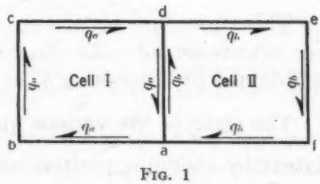

Fig. 1

Hence the following formulas are obtained:

$$q_a \Sigma \frac{\Delta L}{t} \begin{bmatrix} a \to b \\ b \to c \\ c \to d \end{bmatrix} + (q_a - q_b) \frac{\Delta L}{t} [d \to a] - \Sigma q_1 \frac{\Delta L}{t} \text{ (for cell I)} = 0 \ldots (4a)$$

and

$$q_b \Sigma \frac{\Delta L}{t} \begin{bmatrix} d \to e \\ e \to f \\ f \to a \end{bmatrix} + (q_b - q_a) \frac{\Delta L}{t} [d \to a] - \Sigma q_1 \frac{\Delta L}{t} \text{ (for cell II)} = 0 \ldots (4b)$$

in which the symbol [a → b], etc., denotes reference to the member extending from point a, Fig. 1, to point b.

If by symmetry, or by judicious choice of the assumed shear flow, q_a or q_b is equal to zero, Eqs. 4 would be reduced to a single equation:

$$q_a \text{ or } q_b = \frac{\Sigma q_1 \frac{\Delta L}{t}}{\Sigma \frac{\Delta L}{t}} \quad \ldots\ldots (5)$$

4. From Eqs. 2 and 5, denoting q_a or q_b as q_2, the resultant shear flow at any point is found equal to

$$q_3 = q_1 + q_2 \quad \ldots\ldots (6)$$

Case II. With Torsion at Section

The shear distribution throughout a beam section will be modified from that of case I if the beam is subjected to twisting moments in addition to flexural moments. A solution of this problem may be obtained by a procedure similar to that for case I except that steps 3 and 4 should be modified.

In a continuous structure the angles of twist of any two contiguous cells (see Fig. 1) are equal. Therefore, the left side of Eq. 4*a*, step 3, say, $F(\text{I})$ must be equal to the left side of Eq. 4*b*, say $F(\text{II})$; thus:

$$F(\text{I}) = F(\text{II}) \quad \ldots\ldots (7a)$$

Another equation is obtained from statics by equating the moment of the internal shearing forces with the external twisting moment, M_t, with the result that

$$F(\text{I}) + F(\text{II}) = M_t \quad \ldots\ldots (7b)$$

This procedure gives two equations from which the unknowns q_a and q_b can be determined. The final step giving the resultant shear flow at any point is obtained by combining q_a or q_b with q_1, as indicated in step 4, case I.

The signs of the various quantities $q\frac{\Delta L}{t}$ in Eqs. 4 to 7 may be kept consistent by assigning positive or negative signs to these quantities according to whether they cause clockwise or counterclockwise rotation in a particular cell.

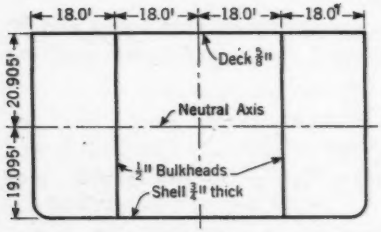

Fig. 2

Example

Fig. 2 shows a cross section of a tanker subjected to a vertical shear of 12,000 kips. The moment of inertia about the neutral axis is 635,000 in^2.-ft^2. It will be assumed that the torque at the section equals zero.

Table 1 indicates, in an orderly manner, the arithmetical computations necessary to obtain the shear distribution for this example. Col. 5, Table 1, gives a set of assumed values of shear flow q_1 computed by Eq. 2*b*. These values of shear flow are plotted in Fig. 3. Because of symmetry the shear

flow at points C and H equals zero; hence the shear flow (in pounds per linear inch) at point B in member BC is obtained directly from Eqs. 2:

$$q_{BC} = \frac{V}{I} z\,\Delta A = \frac{12{,}000{,}000}{635{,}000} 20.905 \times 18 \times \tfrac{5}{8} = -4{,}445$$

and, similarly,

$$q_{GH} = \frac{V}{I} z\,\Delta A = 18.9 \times 19.095 \times 18 \times \tfrac{3}{4} = -4{,}872$$

The sign of shear flow is negative since the rotation is clockwise.

TABLE 1.—SHEAR DISTRIBUTION, TYPICAL MIDSHIP SECTION OF A TANKER

Member[a] (Fig. 3)	ΔA (ft-in.)	z (ft)	$z\,\Delta A$ (ft²-in.)	q_1 (Lb per in.)	q'_1 (Lb per in.)	$\frac{\Delta L}{t}$	$q'_1 \frac{\Delta L}{t}$	q_s shear flow[b]	Shear stress[c]
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
E	....	....	....	0	....	....	....	6,544	13,090
B(E)	10.452	10.452	109.24	− 2,065	− 688	41.81	− 28,765	4,479	8,960
B(C)	11.250	20.905	235.2	− 4,445	....	....	....	−4,445	7,110
B(A)	....	....	....	− 6,509	− 8,732	28.8	−251,482	34	50
A(B)	11.250	20.905	235.2	−10,954	....	....	....	−4,410	7,060
A(D)	15.680	10.452	163.9	−10,954	−13,020	....	−363,649	−4,410	5,880
D	....	....	....	−14,052	....	....	....	−7,508	10,010
F(D)	14.32	9.547	136.7	−11,469	−13,190	25.46	−335,817	−4,925	6,570
F(G)	....	....	....	−11,469	....	....	....	−4,925	6,570
G(F)	13.50	19.095	257.8	− 6,596	− 9,032	....	−216,768	− 52	70
G(H)	13.50	19.095	257.8	− 4,872	....	....	....	−4,872	6,500
G(E)	9.547	9.547	91.2	− 1,723	− 574	38.19	− 21,921	4,820	9,640
Σ						186.19	−1,218.402		

[a] Member or point; the designation B(E), for example, signifies the shear in member BE, Fig. 3, or the properties of member BE. [b] Shear flow, in pounds per inch. [c] Shear stress, in pounds per square inch.

By assuming a shear flow at point E in member BE the shear flow at point B in member BE may be obtained from Eq. 2*b*. Assuming the shear flow at point E equal to zero for convenience, the assumed shear flow is computed as in Cols. 3 and 4, Table 2. Similarly, by use of Eq. 2*b*, the assumed shear flow at each successive point may be computed. This shear flow is shown in Fig. 3.

TABLE 2.—COMPUTATION OF SHEAR FLOW AROUND CELL II, FIG. 3, IN POUNDS PER LINEAR INCH

Shear flow	Member	Computation	Assumed flow	Total flow
(1)	(2)	(3)	(4)	(5)
q_E	....	q_0 in Eq. 2*b*	0	6,544
q_B	BE	18.9×0.5×20.902×20.905×0.5	− 2,065	4,479
q_B	BA	−2,065+(−4,445)	− 6,510	34
q_A	AB	−6,510−(18.9×20.905×18×5/8)	−10,954	−4,410

The assumed shear flow will result in a twist of cell II. Since the angle of twist θ must be zero, a shear flow, acting around the cell, will be introduced as a

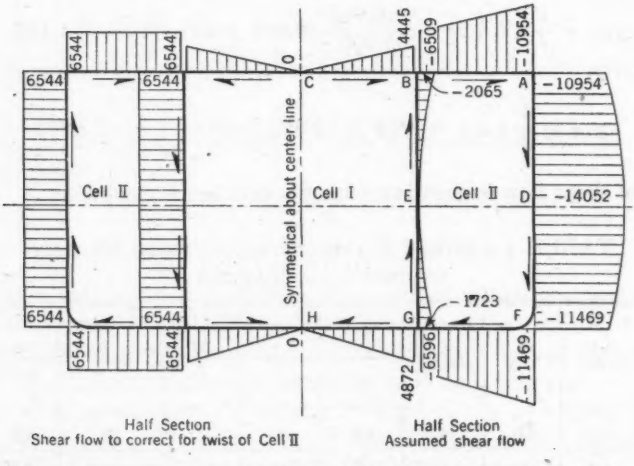

FIG. 3.—COMPUTED SHEARS IN VESSELS WITH LONGITUDINAL BULKHEADS, IN POUNDS PER LINEAR INCH (CLOCKWISE ROTATION IS NEGATIVE)

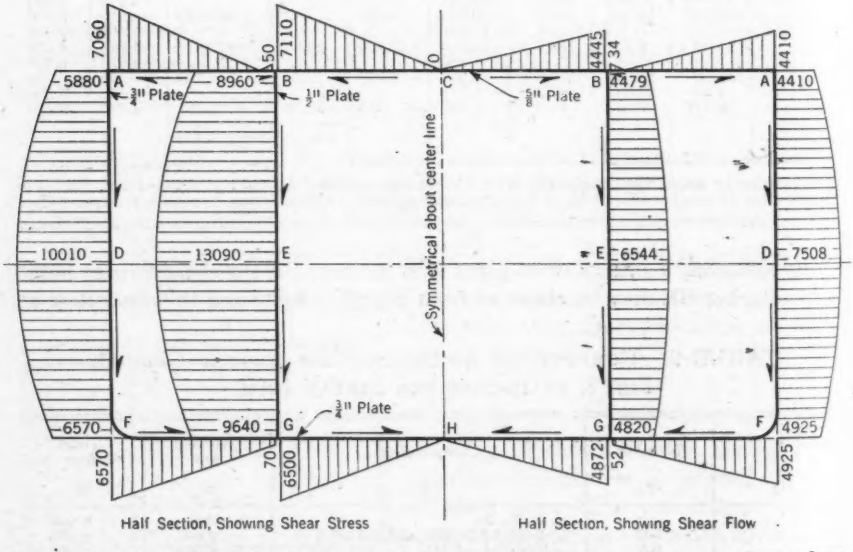

FIG. 4.—FINAL SHEAR STRESSES IN VESSELS WITH LONGITUDINAL BULKHEADS, IN POUNDS PER LINEAR INCH

correction to the assumed shear flow so that the resultant angle of twist will be zero. This can be done by Eq. 5, thus:

$$q_2 = \frac{1{,}218{,}402}{186.19} = 6{,}544 \text{ lb per in.}$$

Applying this constant correction to the assumed shear flow around cell II (Col. 4, Table 2) gives the final shear flow as shown in Col. 5, Table 2, and in Fig. 4. The unit shear stress at any point (in pounds per square inch) is obtained by dividing the shear flow at the point by the thickness of the plate at the point:

$$v_E = \frac{q_E}{t_E} = \frac{6{,}544}{0.5} = 13{,}090$$

$$v_D = \frac{q_D}{t_D} = \frac{7{,}508}{0.75} = 10{,}010$$

If computed by the ordinary formula, the shear stress, $v = \frac{V\,Q}{I\,t}$ and v_E = 11,250 lb per sq in.—or an error in this case on the unsafe side of 16%. The distribution of the shearing stresses is shown in the right half section of Fig. 4.

Summary

The distribution of the shearing stress in a multiple-web beam may be found by the following steps:

1. Assume a statically consistent set of shearing stresses;
2. Find the angular distortion that results from these forces;
3. Introduce a set of shearing forces of such magnitude that the resultant angle of twist will be equal to that known to exist in the structure; and
4. Superpose stresses found in step 3 on those of step 1 to give the desired stress distribution.

Acknowledgments

The procedure for computing shear stress distribution and the particular application of the principles of mechanics to the solution of the problem as outlined in this paper were developed by the writer while engaged in the structural design of concrete barges for the United States Maritime Commission. The writer gratefully acknowledges the help given by members of the staff of the Research Section, Technical Division, of the commission, and particularly the encouragement given by John Vasta, assistant chief of research. Special mention is due C. C. Ragsdale of the Research Section for valuable assistance during the preparation of this paper.

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DISCUSSIONS

RECHARGE AND DEPLETION OF GROUND-WATER SUPPLIES

Discussion

BY CHARLES L. MCGUINNESS

CHARLES L. MCGUINNESS,[1] ESQ.—Mr. Vaubel makes several interesting and useful comments from the standpoint of the development and operation of water supply systems based on wells. He points out that proper well spacing, using the nonequilibrium formula as adapted by Charles V. Theis, is necessary. It might be emphasized that the use of the unmodified Theis formula is quite successful in many parts of the Coastal Plain in Texas and other coastal states, where extensive aquifers are typical. In many places where the aquifers are bounded—or where sources of recharge exist—at short distances from the wells, modification of the formula by intricate mathematical methods may be necessary.

The statement that water of poor chemical quality should be used for purposes in which quality is relatively unimportant deserves great emphasis, especially in the many areas where water of the best quality is limited in quantity. Equally important is the statement concerning the need to consider both ground-water supplies and surface-water supplies in designing each installation.

Mr. Kazmann points out that it was not specified how long was meant by "eventually" in stating that curtailment of irrigation pumping in the High Plains of Texas will eventually be necessary. The length of time depends on the extent to which irrigation pumping, so far entirely uncontrolled, continues to increase. Obviously, adequate legal control by the states of ground-water withdrawals, based on quantitative estimates of the long-term yield obtained by thorough investigation, would go a long way in preventing undue hardship and excessive financial losses in such situations as that in the High Plains.

Mr. Kazmann also points out that the original paper was deficient in not containing a discussion of the effects of technological developments on the availability of ground-water supplies. Such developments in the past, espe-

NOTE.—This paper by Charles L. McGuinness was published in September, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1947, by E. A. Vaubel; and April, 1947, by Raphael G. Kazmann.

[1] Geologist, U. S. Geological Survey, Washington, D. C.

cially the improvement of pumps, have greatly increased the amount of water economically available from wells. They have even led to overpumping in some areas where the aquifers are permeable but the rate of recharge is low. However, with the constant development of improved methods for determining the safe yields of aquifers and with the growth of effective and equitable legal control of ground-water withdrawals, technological progress in methods of extracting ground water may be expected to be entirely beneficial in the future.

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DISCUSSIONS

MODEL STUDY OF BROWN CANYON DEBRIS BARRIER

Discussion

BY KARL J. BERMEL AND ROBERT L. SANKS

KARL J. BERMEL,[16] ASSOC. M. ASCE, AND ROBERT L. SANKS,[17] JUN. ASCE.—The writers wish to express their appreciation of the interest taken by those who submitted discussions; for the valuable contributions, substantiating data, and constructive criticisms of this paper.

Mr. Betts, discussing that phase of the studies dealing with detrital slopes upstream of the barrier, states, "* * * extensive tests covering wide ranges of hydrographs would have added greatly to the information in this field had the project requirements justified the extra cost." Although cost is certainly a consideration, justification of more extensive testing was limited primarily by the lack of sufficient data or knowledge as to the laws governing flood flows and the transport of bed-load material. To obtain this information, aside from the attendant difficulties, is both time consuming and costly and is certainly beyond the scope of these studies; however, until such time as measurements of this type are made and laws established governing both model and prototype behavior, models at best only can be relied upon to indicate general tendencies. This also applies to the first part of the comment by Mr. Amidon, that "* * * it is regrettable that more definite data did not result relating to the movement of debris and to the probable ultimate debris slope to be stabilized by the structure."

As for the latter part of Mr. Amidon's comment, it was previously mentioned in the paper that true stability (ultimate debris slopes) would never be achieved since variations in flow, bed-load size, and amount available for transportation, as well as changes in flow sections, changes of vegetative growth in the channel bottom, and many other factors would influence the debris slopes that would exist after each successive flood. If these factors and the variation of these factors were known, and could be predicted for the future, then model studies

NOTE.—This paper by Karl J. Bermel and Robert L. Sanks was published in May, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1946, by Clifford A. Betts, and Roger E. Amidon; and December, 1946, by Ettore Scimemi, and J. W. Johnson.

[16] Senior Engr., Div. of Mech. Eng., Univ. of Calif., Berkeley, Calif.

[17] Asst. Engr., Univ. of California, Berkeley, Calif.

could be made that would provide information as to the maximum, minimum, or average storage that would attain for the period of time considered.

It is unfortunate that Mr. Amidon did not include more detailed data regarding the measurements of debris slopes above the approximately one hundred existing structures. These measurements cover a sufficient range of original channel slopes so that it might be possible to determine if the debris slopes as measured vary as a function of the original channel slopes. In the paper, the final detrital slopes, as achieved in the model, were compared with the original channel slopes; however, the findings of the model studies indicate that the detrital slopes vary as a function of the flow, the size and gradation of the bed-load material, and the amount of bed load transported or available for transport.

For the experiments performed to determine the extent of scour, the maximum depth of scour, and the distance downstream to the point of maximum depth of scour, the authors used the experimental method as suggested by Mr. Scimemi, with the exception that a graded bed-load material was used instead of material successively smaller in grain size. This material was the same as that used for the deposition studies upstream of the barrier; that is, it varied from 0.1 mm to 15 mm in size. The discrepancy of 3 ft less depth of scour, as compared to the experiments performed at the University of Padua, Italy, is no doubt due to the retention of the larger sized particles. Whereas the latter experiments are more exact as to the limiting depth of scour, the use of the bed-load material was considered more indicative of the conditions that would exist after bed-load material was transported over the barrier, which would be the conditions of operation for the greatest part of the life of the structure.

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DISCUSSIONS

TRUCK SPEED AND TIME LOSS ON GRADES

Discussion

BY J. W. STEVENS

J. W. STEVENS,[12] Assoc. M. ASCE.—The discussions indicate an opinion which is evidently held by many private drivers: That trucks on hills delay traffic to an annoying extent. The writer heartily agrees with the comments pointing out the need for passing lanes on many existing highway grades. It is easy to determine whether there is serious need for a passing lane on an existing highway grade merely by observation of traffic in the field and without analysis of the profile and possible truck speed.

Passing lanes should be provided in the original design, instead of waiting until the pavement is built before determination of their need. Therefore, the application of the speed and time loss studies is most valuable in the general design of a project on new location. In many cases the need for passing lanes will be evident without detailed study, and the necessity for such study will be reduced to the doubtful or borderline cases.

Special emphasis should be given to the limitation of the conclusion at the end of Section 16 (which applies strictly to overpass grades in otherwise flat terrain) as pointed out by Mr. Jones. It is probable that passing lanes might be amply justified on fairly steep grades involving rises of the order of 50 ft on highways where a rise of 25 ft would not require an extra lane.

Mr. Ogle's discussion indicates that the question of permitting automobiles to pass slow-moving trucks is only part of the designer's problem. The need for grade design to provide for efficient operation of truck traffic is increasing, and will continue to increase as the proportion of truck traffic grows.

The writer feels that the published discussions are valuable for their additions to the ideas covered in the paper, and wishes to express appreciation for these discussions.

Attention should be called again to the possibility of combining arithmetical operations to shorten the computations, as mentioned in the first sentence in Section 11. Any user of the method will find additional short cuts particularly

NOTE.—This paper by J. W. Stevens was published in September, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1946, by Jonathan Jones; and March, 1947, by Elliott J. Dent, and F. B. Ogle.

[12] Bldg. Insp. Engr., City of Amarillo, Amarillo, Tex.

suited to his procedure. It is interesting to note that both the shape and height of the velocity curve in Fig. 3 show considerable resemblance to the profile curve. This resemblance, of course, depends on the scales of the two curves and on the applied power of the vehicle, and possibly is purely coincidental. However, remarkable similarity of the velocity and profile curves was noted in all cases covered by Fig. 5. It is conceivable that the similarity of these curves might be established for short sections of profile in many cases, so that the velocity curve could be drawn with satisfactory accuracy by inspection of the profile, thus eliminating computations.

It is hoped that the remarks in Section 14 will discourage no one from making full use of graphic methods in speed and time loss problems. The writer's attention has been called to the possibility of graphic solutions shorter than those of which he was aware when the paper was prepared. The graphic methods for time loss determination might well form the subject of a valuable paper for the benefit of engineers concerned with this phase of highway design.

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DISCUSSIONS

DRAWDOWN TEST TO DETERMINE EFFECTIVE RADIUS OF ARTESIAN WELL

Discussion

BY C. E. JACOB

C. E. JACOB,[15] Assoc. M. ASCE.—Although few in number and brief, the discussions have added much to the paper and, moreover, have suggested the direction that further work might profitably take. The writer wishes to thank those who have contributed.

The question raised by Mr. Boulton is a pertinent one—regarding the need for evidence to support the assumption that the compressibility of the aquifer has the same value near to, and far from, the pumped well despite the wide range of pressure release. The writer knows of no close observations of long-period decline under constant and continuous pumping that might clarify this problem. Of course, it is to be expected that an unconsolidated sand would have a variable compressibility, depending on the rate and magnitude of the loading occasioned by the release of pressure. Moreover, sight should not be lost of the fact that the flexural rigidity of the overlying beds complicates the problem, especially in the immediate vicinity of pumped wells tapping deep aquifers. Fortunately, however, as long as the compressibility (thus modified) may be assumed reasonably constant in time at a given distance the uncertainties arising from ignorance of its absolute value are reflected only in the degree of approximation of r_w. Actually, in predictions of future drawdown based on the theory of an elastic aquifer, the product $S r^2_w$ is used. This product is determined empirically and it is not necessary to break it down into S and r_w, except to compare different wells. If S varies with t, for one reason or another, then some other theory must be used or the elastic theory must be modified.

Mr. Boulton's estimate of 10.5 ft for the friction and velocity head losses accompanying the upward flow inside the well casing of the Bethpage well narrows down the well loss to that part that may be termed "screen loss." His calculation of the "convergence loss" under laminar flow into an assumed system of vertical slots shows that such loss would be quite small. Actually

NOTE.—This paper by C. E. Jacob was published in May, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1946, by N. S. Boulton; December, 1946, by Carl Rohwer; and February, 1947, by R. M. Leggette, and M. R. Lewis.

[15] Senior Hydr. Engr., Water Resources Branch, U. S. Geological Survey, Washington, D. C.

the openings in the screen are in the form of a helix, but the "convergence loss" there should be of the same or even of a smaller order of magnitude. Clogging of the slots is a distinct possibility, as pointed out by Mr. Boulton. Furthermore, departure from laminar flow may begin as the water passes through the screen openings, especially if they are clogged.

By calculation Mr. Rohwer shows that the combined relative compression of the aquifer near Meadville, Pa., and its contained water is only about 1 in 5,000 for a 23-ft drop in head. He states:

> "In view of the large unavoidable errors involved in other measurements it seems that this factor [compressibility of water] could well be neglected. The same may be stated of the compression of the aquifer."

Perhaps Mr. Rohwer has in mind estimating the ultimate or steady-state discharge of wells in a limited aquifer, in which case the volume of water derived from artesian storage might soon become small by comparison with the volume drawn through the aquifer from an outer boundary. However, until that steady state of flow is established, water is withdrawn from storage through its elastic expansion and through the concomitant compression or compaction of the aquifer. Indeed, in extensive and deep-lying aquifers such as the Dakota sandstone, the supply may be furnished entirely from storage for decades. Because of the tremendous volume of water in such an aquifer and because of the sizable lowering of head that may be produced, the total volume of water withdrawn from storage through wells may itself reach an astounding magnitude. In any event, however insignificant this factor may appear, the study of the transient behavior of an elastic aquifer, on which the paper is based, requires an appraisal of its magnitude.

With reference to the large well loss of 28.5 ft at 1,350 gal per min in the well near Meadville, Pa., the suggestion is offered by Mr. Rohwer that the screen may be encrusted or improperly perforated. As the well was newly constructed when tested, it would seem that sand packing of the gravel envelope and of the screen might be the explanation. The formation is a uniform fine sand—difficult to handle in drilling and developing a well.

Mr. Rohwer adds a valuable point in remarking on the unavoidable magnification of errors through elimination of variables by subtraction from the series of equations. He justifiably emphasizes the need for care in treating such data as those presented in the paper. Whereas from a theoretical standpoint, with ideal data, the procedure outlined in the paper is sound; in practice it needs modification. Higher precision of measurement may warrant the assumption that the drawdown obeys the law $s_w = BQ + CQ^n$, introducing a third unknown, the exponent n (<2), to be determined by trial-and-error computation, or by graphical procedure together with the coefficients B and C.

An important point is raised by Mr. Leggette—that the advantages of gravel packing or developing a well may be offset by poor design or improper choice of screen. The writer feels that amassing empirical values of C and of r_w, together with pertinent data on the details of design and construction of wells, may eventually make possible the accurate appraisal of these various factors. The selection of screen type, slot opening, and gravel size—and even

the determination of whether or not a gravel envelope is required—may be lifted from the realm of guesswork to a rational plane through the future study of existing and newly constructed wells and through the measurement of their characteristics of performance, due consideration being given to the transient behavior of the aquifer.

In emphasizing the magnitude of well losses, condemnation of the well driller is not intended, for much of the friction loss in and near a well is unavoidable and will never be entirely eliminated. Nevertheless, it behooves the engineer and the well-drilling contractor alike to strive for as efficient design and construction as possible to meet the stringencies of economic demands. The points summarized by Mr. Leggette thus are objectives toward which progress should be made.

Mr. Lewis suggests that similar analyses be made for unconfined flow under simple water-table conditions, and for confined flow in which recharge from the soil surface occurs through a relatively impermeable confining bed. To the writer's knowledge a satisfactory analysis of nonsteady unconfined flow has not been given. Even in the case where the storage coefficient (S, ultimately approaching "specific yield") is constant, there are insuperable difficulties. Only by analogy to confined flow, and then in cases where the maximum drawdown is but a small fraction of the initial depth of flow, has an approximate solution been obtained. It may be stated, however, that even in the absence of well losses the specific capacity of a water-table well would vary with the discharge, the curve of drawdown versus discharge at constant time being a parabola under certain approximative assumptions. Further work should be done on this problem, both in the laboratory and in the field.

A solution has been given for the nonsteady radial flow toward a steadily discharging well in a leaky confined aquifer.[16] The leakage is assumed proportional to the drawdown. Whether this is exactly the condition Mr. Lewis has in mind is not known, but in the early phase of a transient state such a system acts like an ideal elastic aquifer without leakage. Accordingly, a short multiple-step drawdown test could be analyzed under those conditions on the basis of the elastic theory, although long-term predictions would consider the leakage.

[16] "Radial Flow in a Leaky Artesian Aquifer," by C. E. Jacob, *Transactions*, Am. Geophysical Union, Vol. 27, 1946, Pt. II, pp. 198–205.

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DISCUSSIONS

RIGID-FRAME STRUCTURES SUBJECT TO NONUNIFORM THERMAL ACTION

Discussion

BY CARL C. H. TOMMERUP

CARL C. H. TOMMERUP,[13] M. ASCE.—Many new ideas, twists, and short cuts have been advanced in the interesting discussions, for which the writer wishes to express his gratitude.

Some of the discussers criticize the fact that the computations in the paper are not sufficiently accurate and that various approximations are made. Be that as it may. A far greater number of engineers have criticized the paper on the ground that the computations were needlessly exact.

Enlightening as the discussions are, the writer is greatly disappointed that the discussers submit so little actual field data on similar problems. By actual data are meant observations, tests, and unusual design features. It seems reasonable to assume that many engineers have in the past been confronted with the same type of problem in one form or another.

The paper was submitted in the hope of extracting replies from engineers who not only had dealt with similar problems, but who had also collected experimental data on uneven heat transfer through structural members. A large amount of experimental work in this field is in order, and it is sincerely hoped that engineers who have already compiled such data will make them available to the profession. It is equally desirable that properly equipped institutions conduct tests along these lines in order to eliminate much of the present guesswork in structural thermal computations.

NOTE.—This paper by Carl C. H. Tommerup was published in June, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1946, by I. Oesterblom, and Frank R. Higley; January, 1947, by A. L. Miller; March, 1947, by Charles O. Boynton and June, 1947, by William A. Conwell.

[13] Senior Structural Engr., Pacific Islands Engrs., Guam, Marianas Islands.

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DISCUSSIONS

MATRIX ANALYSIS OF CONTINUOUS BEAMS

Discussion

BY SVEN OLOF ASPLUND, AND STANLEY U. BENSCOTER

SVEN OLOF ASPLUND,[34] M. ASCE.—How matrices may be applied in the analysis of continuous beams is shown very constructively by the author, who uses matrix methods to demonstrate that the result of the iterative moment distribution method developed by Hardy Cross, M. ASCE, approaches that of the exact classical solution. Obviously in recognition of the generality of matrix methods, Mr. Benscoter suggests the possibility of useful application of matrices to the analysis of other statically indeterminate structures. The writer has actually employed matrices in the analysis of pile groups. In such a case the matrices originate from the vectors used in determining the acting forces and their positions.

In European practice[35] pile groups have for many years[36] been calculated according to the elastic theory. For plane pile groups the calculations are easily accomplished,[37] but for spatial pile groups the computation will become almost prohibitive if matrices are not employed.

The external forces acting upon the rigid pile pier cause a small translation, defined by a vector $\mathbf{t}$ (Fig. 14), and a small rotation, r, about an axis of rotation, $\mathbf{r}$, through the origin O. The pile heads, h_j, in the pier are located by the vectors $O\,h_j = \mathbf{b}_j$, and the total movement of the pile head is given by $\mathbf{t} + [\mathbf{r}\,\mathbf{b}_j]$ (brackets indicate vector products). The projection of this movement upon the unit vector $\mathbf{p}_j$ along the pile axis is the scalar product $\mathbf{p}_j(\mathbf{t} + [\mathbf{r}\,\mathbf{b}_j]) = \mathbf{p}_j\,\mathbf{t} + [\mathbf{b}_j\,\mathbf{p}_j]\,\mathbf{r}$. The pile shortens by this length, causing an elastic pile reaction:

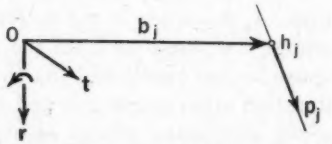

FIG. 14.—VECTORS REPRESENTING PILE FORCES AND MOVEMENTS OF PILE GROUP

$$\lambda_j\,\mathbf{p}_j\,(\mathbf{p}_j\,\mathbf{t} + [\mathbf{h}_j\,\mathbf{p}_j]\,\mathbf{r}) \qquad (168)$$

NOTE.—This paper by Stanley U. Benscoter was published in October, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1947, by I. Oesterblom, and Harris Solman; May, 1947, by Rufus Oldenburger, Horace A. Johnson, Lloyd T. Cheney, and Paul W. Norton; and June, 1947, by A. Floris.

[34] Docent, Bridge Eng. Dept., Royal Inst. of Tech., Stockholm, Sweden.

[35] "Der Kampf des Ingenieurs gegen Erde und Wasser im Grundbau," by A. Agatz, Springer, Berlin, 1936.

[36] "Teori för Grundpålningar," by Per Gullander, Norstedt, Stockholm, 1914.

[37] "Beregning av Paelevaerker," by Chr. Nøkkentved, Gad, Copenhagen, 1923.

Aside from the pile reactions, all other forces that act upon the pier may be reduced to a force, **f**, and a moment, **m**, about O. The equilibrium of the pier demands that

$$\mathbf{f} = \Sigma\lambda\, \mathbf{p}\,(\mathbf{p}\,\mathbf{t} + [\mathbf{b}\,\mathbf{p}]\,\mathbf{r}) = \mathbf{T}\,\mathbf{t} + \mathbf{U}\,\mathbf{r} \quad \ldots\ldots\ldots\ldots (169a)$$

$$\mathbf{m} = \Sigma\lambda\, [\mathbf{b}\,\mathbf{p}] \times \mathbf{p}\,\mathbf{t} + \Sigma\lambda\, [\mathbf{b}\,\mathbf{p}] \times [\mathbf{b}\,\mathbf{p}]\,\mathbf{r} = \mathbf{U}'\,\mathbf{t} + \mathbf{V}\,\mathbf{r} \quad \ldots\ldots (169b)$$

or, in Mr. Benscoter's matrix notation,

$$[f] = [T]\,[t] + [U]\,[r] \quad \ldots\ldots\ldots\ldots\ldots\ldots (170a)$$

$$[m] = [U]'\,[t] + [V]\,[r] \quad \ldots\ldots\ldots\ldots\ldots\ldots (170b)$$

This matrix notation appears to the writer to be unnecessarily laborious. Henry Margenau and G. M. Murphy[38] and Harold Jeffreys[39] use bold face letters to denote matrices, whereas Richard Courant and David Hilbert[40] use ordinary letters:

$$\left.\begin{aligned} f &= T\,t + U\,r \\ m &= U'\,t + V\,r \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (171)$$

The symbols T, U, and V may be recognized as third-order square matrices with elements $T_{kl} = \Sigma\lambda\, p_k\, p_l$, $U_{kl} = \Sigma\lambda\, p_k\, [\mathbf{b}\,\mathbf{p}]_l$, etc. The symbol U' is the transpose of U. The solution of Eq. 171 is easily found to be

$$\left.\begin{aligned} t &= T^{-1}{}_0\, f - T^{-1}{}_0\, U\, V^{-1}\, m \\ r &= -\, V^{-1}{}_0\, U'\, T^{-1} f + V^{-1}{}_0\, m \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (172)$$

in which $T_0 = T - U\,V^{-1}\,U'$ and $V_0 = V - U'\,T^{-1}\,U$. Substitution of Eqs. 172 in Eq. 168 yields the force in each pile. The numerical computations may be systematically and effectively carried through according to Eqs. 172.

It is often possible to add to an actual pile group one or more fictive piles in such a manner that the resulting pile group may be more expediently analyzed —for example, by resulting symmetry. The forces that the original external forces **f**, **m** induce in the fictive piles are evaluated and added to **f**, **m** to neutralize the reactions from the fictive piles upon the pier. The added forces cause further reactions from the fictive piles which are again added, etc. This iteration often converges, and it is possible by summation of geometric matrix series (Neumann series) as explained by the author to show that the final result is identical to a direct solution of the actual original pile group in accordance with Eq. 172.

STANLEY U. BENSCOTER,[41] ASSOC. M. ASCE.—The discussions contain several interesting additions to the paper. Several writers question the usefulness of matrix methods in more complicated problems such as the analysis of frameworks with joint translations. In such problems matrix algebra provides a convenient process of formulating and defining the various calculating

[38] "The Mathematics of Physics and Chemistry," by Henry Margenau and G. M. Murphy, D. Van Nostrand Co., Inc., New York, N. Y., 1943.

[39] "Methods of Mathematical Physics," by Harold Jeffreys, Cambridge Univ. Press, Cambridge, England, 1946.

[40] "Methoden der Mathematischen Physik," by Richard Courant and David Hilbert, Springer, Berlin, 1931.

[41] Structural Engr., Structures Div., N.A.C.A., Langley Memorial Aeronautical Laboratory, Langley Field, Va.

procedures that may be used. The best method of calculation will depend on such factors as the type of computing device to be used, the size of the matrix of coefficients, and the number of times that the equations are to be solved (number of loading conditions). One should not expect to analyze an enormous framework on a slide rule any more than one would expect to dig an enormous excavation with a hand spade. Just as large electrical or mechanical shovels are used for large excavation jobs, so must one use large electrical or mechanical computing machines for analyzing large frameworks.

For solving a general system of equations, evaluating a determinant, or computing a reciprocal matrix, on a hand-operated calculating machine, the method of P. D. Crout[42] is the most efficient that is known to the writer. This method is being adopted quite generally throughout the United States in design and research work. More powerful calculating machines have been developed in recent years such as the punch card machine and the sequence-controlled electronic or electrical relay machines. These machines operate automatically with iteration procedures and perform successive matrix multiplications conveniently. They solve as many as one hundred simultaneous equations and provide the only real answer to the civil engineers' need for a method of analyzing large frameworks. Matrix algebra and the modern calculating machine are complementary tools for use in the solution of physical problems for which solutions in terms of analytical functions cannot be obtained. Both of those tools must be developed simultaneously to obtain the maximum benefit from either.

The historical note by Mr. Oesterblom is interesting. His question concerning the possibility of representing the reciprocal of the stiffness matrix for a framework by a power series can be answered by stating that the reciprocal of any square matrix can be represented by a power series if the determinant of the matrix does not vanish. Re-analysis of a structure for loading changes involves the use of the same stiffness matrix and, hence, is not a serious problem in design. Re-analysis for framework dimension changes is an entirely different matter because this procedure involves changes in the stiffness matrix, which require a new computation of the reciprocal matrix. However, if the changes in stiffness values are small relative to the originally assumed stiffnesses, a convenient method of obtaining the new reciprocal matrix can be developed.

Assume that the original stiffnesses give the square matrix $[K]$ as the coefficient matrix in the equation of three slopes (Eq. 67). From a stress analysis it is found that changes in the dimensions of the members are required. The new stiffness matrix becomes $[K + \Delta K]$. The square matrix $[\Delta K]$ contains small numbers representing the changes in stiffness of the members. Define the square matrix $[\delta]$ by

$$[\delta] = [K]^{-1}[\Delta K] \quad \ldots\ldots\ldots\ldots\ldots\ldots (173)$$

Eq. 173 can be computed immediately since the reciprocal matrix $[K]^{-1}$ is known. The new stiffness matrix may be written as

$$[K + \Delta K] = [K][I + \delta] \quad \ldots\ldots\ldots\ldots\ldots\ldots (174)$$

[42] "A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients," by P. D. Crout, *Transactions*, A.I.E.E., Vol. 60, 1941, pp. 1235–1240.

The reciprocal may be written as

$$[K + \Delta K]^{-1} = \{[I] - [\delta] + [\delta]^2 - \cdots\}[K]^{-1} \ldots\ldots\ldots (175)$$

Since the elements of $[\delta]$ are small relative to unity, the second and higher degree terms in the series may be neglected:

$$[K + \Delta K]^{-1} = [I - \delta][K]^{-1} \ldots\ldots\ldots\ldots\ldots (176)$$

Thus, the original reciprocal matrix is easily converted into the new reciprocal matrix. The development of practical design procedures on the basis of this theory is a worthwhile objective for future research study.

The formulas for moments given by Mr. Solman form an important contribution to the analysis of continuous beams. Since there are a number of other problems in both civil and aeronautical design which lead to a system of equations having a continuant matrix, the possibility of developing such formulas is of considerable interest. The formulas become especially useful with symmetrical structures. The simple formulas of Eqs. 132 and 133 may be used to analyze symmetrical beams having as many as six spans. This classification includes a large part of all highway bridges.

It is clearly shown by the formulas of Mr. Solman that the solution for moments is dependent only on the fixed-end moments and the primary and secondary distribution factors. The convenience of introducing product factors $p_{ij} = q_{ij}\, q_{ji}$ is also evident. The significant feature of these scalar formulas, from a general mathematical viewpoint, lies in the fact that they arise from formulas for the elements of the reciprocal of the matrix $\{[I] - [Q]'\}$. The development of the formulas, as given by Mr. Solman, is so brief as to be difficult for a reader to follow. The development employs certain special properties of a continuant matrix and its adjoint that are not known to the majority of readers.

The special case of two effective joints has been noted by Hardy Cross,[43] M. ASCE. In this case the power series of matrices degenerates into a scalar geometric series and direct formulas for the solution are readily developed.

The writer is in agreement with the comments of Professor Oldenburger. It was the classical paper of Professor Oldenburger[9] that inspired the writer to take up the study of matrix algebra. Studies of the convergence of the power series, which defines moment distribution, lead to new methods of computing the critical, or characteristic, values of buckling or vibratory loads. The problem of framework buckling is in serious need of research attention so that more rational methods of design may be developed.

Mr. Johnson has illustrated the use of the Cramer rule in solving a system of equations. The formula for computing a reciprocal matrix as given in Eq. 33, without proof, would be developed by using this rule. The Cramer rule is very useful for solving either two or three simultaneous equations when the equations are to be solved only once (one loading condition). For more than three equations other methods should be used even when solving the equations only once.

[43] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 125.

Professor Cheney has illustrated the standard engineering textbook method of successive elimination of unknowns. The evaluation of a 4×4 determinant, as given in Eq. 148, is correct. However, the evaluation of a 6×6 determinant, as shown in Eqs. 150 and 151, is erroneous. Unfortunately, the task cannot be simplified in this manner.

Mr. Norton has illustrated in Fig. 11 the association that exists between the stiffness matrix for a continuous beam and the structure itself. He has also explained in detail the Gauss method of pivotal condensation[44] for evaluating determinants. By using the Gaussian method it is found that the elements of the adjoint of a continuant matrix may be expressed as given by Eq. 160. This development is of considerable importance for a number of different structural problems. The pattern of formation of the adjoint is clear and convenient to use. The calculations are easily performed when the factors of an element are expressed as continued fractions. Consider the element of the first row and first column of the matrix of Eq. 160. Apply the formula to the matrix $\{[I] - [Q]'\}$ of the fifth order; thus:

$$A_{11} = a_{22}\, a'_{33}\, a''_{44}\, a'''_{55}$$

$$= 1 \times (1 - p_{23}) \left(1 - \frac{p_{34}}{1 - p_{23}}\right) \left[1 - \frac{p_{45}}{1 - \dfrac{p_{34}}{1 - p_{23}}}\right] \quad \ldots\ldots (177)$$

in which $p_{ij} = q_{ij}\, q_{ji}$. If the multiplication indicated in Eq. 177 is performed, the resulting formula will be found to correspond to the first element of the matrix of Eq. 146, as given by Mr. Solman. The determinant of the matrix may also be expressed as a product of continued fractions.

Mr. Floris mentions the possibility of negative stiffness values when shearing deformations are considered. There is some doubt as to whether rotational stiffnesses are affected by shearing deformation—whether they are or are not is dependent on the definition of stiffness. There are at least two possible methods. The problem is in need of still further attention by research workers. It is possible that only the translational stiffnesses (which are not mentioned in the paper) are affected. The complicating effect of considering shear deformation is that it brings about a discontinuity in the slope of the elastic curve at points of application of concentrated loads or reactions. In the case of an applied load the magnitude of the discontinuity is known, whereas in the case of a reaction the magnitude is unknown.

The simple rule for inverting a 2×2 matrix, as noted by Mr. Floris, is very useful. The writer has used it many times in design and research work. The adjoint is formed by interchanging the diagonal elements of the matrix and by merely changing the signs of the nondiagonal elements.

Mr. Asplund has illustrated the use of matrices in the solution of a very interesting problem. The analysis of pile groups has received little attention in American literature. The plane problem was treated by C. P. Vetter,[45] M. ASCE. The space group with arbitrary pile batter, pile length, and head position is a more complex problem.

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[44] "Determinants and Matrices," by A. C. Aitken, Interscience Publishers, Inc., New York, N. Y., 1939, p. 45.

[45] "Design of Pile Foundations," by C. P. Vetter, *Transactions*, ASCE, Vol. 104, 1939, pp. 758–811.

Since the development of the formulas for the elements of the matrices has been given by Mr. Asplund in the language of vector algebra, it may be interesting to show the same development in the language of matrix algebra. This statement is especially true since vector products arise in the analysis. Vectors in vector notation will be equated to vectors (column vectors or row vectors) in matrix notation. The equality sign must be understood to mean equality of the components of the vectors.

The pier translation is indicated by the vector:

$$\mathbf{t} = [t]' = [t_1\, t_2\, t_3] \quad \ldots\ldots (178)$$

The pier rotation is indicated by

$$\mathbf{r} = [r]' = [r_1\, r_2\, r_3] \quad \ldots\ldots (179)$$

In developing relationships for a single pile, the subscript, j, will be omitted for convenience. The vector from the origin to the pile head is given by

$$\mathbf{b} = [b]' = [b_1\, b_2\, b_3] \quad \ldots\ldots (180)$$

To define vector products in matrix notation, it is necessary to introduce two skew-symmetric matrices that may be constructed from the components of the vectors $[r]$ and $[b]$:

$$[R] = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \quad \ldots\ldots (181)$$

$$[B] = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \quad \ldots\ldots (182)$$

A skew-symmetric matrix is one that is equal to the negative of its transpose:

$$[R] = -[R]'; \quad [B] = -[B]' \quad \ldots\ldots (183)$$

A vector product may now be illustrated; thus:

$$\mathbf{r} \times \mathbf{b} = [R][b] = -[B][r] \quad \ldots\ldots (184)$$

The resultant translation of the pile head may then be expressed by

$$\mathbf{t} + \mathbf{r} \times \mathbf{b} = [t] + [R][b] = [t] - [B][r] \quad \ldots\ldots (185)$$

A unit vector along the pile axis is indicated by

$$\mathbf{p} = [p]' = [p_1\, p_2\, p_3] \quad \ldots\ldots (186)$$

The components of this vector are the direction cosines of the pile axis. The shortening of the pile is given by

$$\mathbf{p} \times (\mathbf{t} + \mathbf{r} \times \mathbf{b}) = [p]'\{[t] - [B][r]\} \quad \ldots\ldots (187)$$

The quantity on either side of Eq. 187 is a scalar number. When multiplied by an axial stiffness constant, λ, the magnitude of the pile reaction force is obtained. If this value is then multiplied by the vector, $\mathbf{p}$, the reaction will

be expressed as a vector:

$$\lambda \mathbf{p}\,[\mathbf{p}\cdot\mathbf{t} + \mathbf{p}\cdot(\mathbf{r}\times\mathbf{b})] = \lambda\,[p]\{[p]'[t] - [p]'[B][r]\} \quad \ldots\ldots(188)$$

or

$$\lambda \mathbf{p}\,[\mathbf{p}\cdot\mathbf{t} + \mathbf{p}\cdot(\mathbf{r}\times\mathbf{b})] = \lambda\,\{[P][t] - [P][B][r]\} \quad \ldots\ldots(189)$$

in which

$$[P] = [p][p]' = \begin{bmatrix} p^2_1 & p_1 p_2 & p_1 p_3 \\ p_2 p_1 & p^2_2 & p_2 p_3 \\ p_3 p_1 & p_3 p_2 & p^2_3 \end{bmatrix} \quad \ldots\ldots(190)$$

In the analysis given by Mr. Asplund five of the six stiffnesses that may be defined for the pile have been regarded as negligible relative to the axial stiffnesses. The neglected stiffnesses are the bending translational stiffnesses, the bending rotational stiffnesses, and the twisting rotational stiffness. The assumption is most directly applicable to piles driven through soft ground to bedrock. In the case of friction piles the foundation itself participates to a considerable extent in the action to complicate the problem. The assumption of a constant of proportionality between reaction force and displacement function should be replaced by a more general functional relationship, as determined by an appropriate influence function. To the equilibrium solution given by Mr. Asplund should be added a linear combination of a finite set of self-equilibrating characteristic distributions to allow for the foundation effect. The mathematical difficulties of obtaining such a solution are, at present, insurmountable. The theory of Mr. Asplund is probably as good, in most cases, as the commonly used assumption of planar distribution of reaction beneath piers, walls, dams, etc., without piles.

The summation of all reaction forces and their moments must equal the applied force, $[f]$, and the applied moment, $[m]$. The subscript, j, will now be introduced where necessary for the summation:

$$\Sigma\lambda_j\,\{[p_i][t] - [P_i][B_i][r]\} = [f] \quad \ldots\ldots(191)$$

$$\Sigma\lambda_i\,[B_i]\{[P_i][t] - [P_i][B_i][r]\} = [m] \quad \ldots\ldots(192)$$

Substituting Eq. 183 in Eq. 192 gives

$$\Sigma\lambda_i\,[B_i]'\,\{-[P_i][t] + [P_i][B_i][r]\} = [m] \quad \ldots\ldots(193)$$

The following matrices may be introduced:

$$[T] = \Sigma\lambda_i\,[P_i] \quad \ldots\ldots(194)$$

$$[U] = -\Sigma\lambda_i\,[P_i][B_i] \quad \ldots\ldots(195)$$

$$[V] = \Sigma_{\lambda i}\,[B_i]'\,[P_i][B_i] \quad \ldots\ldots(196)$$

These matrices were introduced by Mr. Asplund to give the two matrix formulas (Eq. 170*a* and Eq. 170*b*). The two equations may be written as a single equation, thus:

$$\begin{bmatrix} T & U \\ U' & V \end{bmatrix}\begin{bmatrix} t \\ r \end{bmatrix} = \begin{bmatrix} f \\ m \end{bmatrix} \quad \ldots\ldots(197)$$

The matrix equation (Eq. 197) represents six linear algebraic equations that

must be solved simultaneously. In the case of general 6×6 matrices the method of Professor Crout[37] will usually be found to be the most rapid on a hand-operated calculating machine. Partitioning of the matrices, as illustrated by Mr. Asplund, becomes advantageous when the submatrices $[T]$, $[U]$, and $[V]$ have special simplified forms. A study of these simplified forms for various typical pile arrangements would be an interesting research problem. It should be noted that the representation of matrix division by fractions, as shown in Eq. 172, is ambiguous. It does not indicate the correct order of the matrix multiplication.

The discussions have revealed a number of additional suggestions for the use of determinants and matrices in the solution of linear equations of structures. The paper has introduced only the simplest concepts of matrix algebra. It is to be hoped that there will be continued advancement in the application of matrix theory to structural analysis problems.

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DISCUSSIONS

CORRECTION OF TAILWATER EROSION AT PRAIRIE DU SAC DAM

Discussion

BY C. N. WARD AND HENRY J. HUNT

C. N. WARD,[18] M. ASCE, AND HENRY J. HUNT, ESQ.[19]—A number of pertinent and interesting factors associated with erosion below dams were emphasized by the various discussions of the paper. The writers appreciate the favorable comments as to the final plans of improvement which were adopted and as to the use of models in such studies.

At the time the Prairie du Sac Dam, Wisconsin, was designed and constructed, 1910-1914, little was known about drop in tailwater due to the movement of stream bed material. Inasmuch as this dam design was unique in many ways, and a departure from the standard practice, a careful check has been maintained on the structure from year to year. From the hourly plant records many interesting curves have been prepared, of which Fig. 8 is highly significant. For example, to understand the curves marked C_1, let a, b, c, d, and e be the annual tailwater elevations for successive years. Then C_1, the progressive mean for the middle year (in which the actual value is C), is expressed as:

$$C_1 = \frac{a + 4b + 6c + 4d + e}{16} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The general trend of the drop in the lowest tailwater experienced each year since 1915 is shown on the lower curve. A similar curve for the mean of the twelve monthly low tailwater levels for each year is also shown for the same period, and the mean monthly tailwater levels for the period 1929 to 1947.

The general trend, as shown by Fig. 8, is toward a stabilized tailwater level. Several conditions may be mentioned that probably have a bearing on this stabilization. Originally, the slope of the river for some distance downstream was about 2.5 ft per mile, whereas today the slope in the first 7 miles is

NOTE.—This paper by C. N. Ward and Henry J. Hunt was published in October, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1947, by George P. Steinmetz, Louis M. Laushey, Abraham Streiff, and R. H. Keays.

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about 1.5 ft per mile. For corresponding flows, the channel width is greater and the area larger now (1947) than in 1915 because of the reduced slope and velocity. The amount of material moved out of the first 7 miles of river is estimated at 11,000,000 cu yd. A large part of this material was from the first 2 miles below the dam, in which reach one large island and several other smaller areas were carried downstream. From a visual inspection of the river bottom downstream from the dam, during the early morning hours when the flow is especially low, it is obvious that several controls play their part in the stabilization of the tailwater level. For instance, there are two highway bridges and one railroad bridge, a few old government wing dams, and coarse gravel banks, which act as controls.

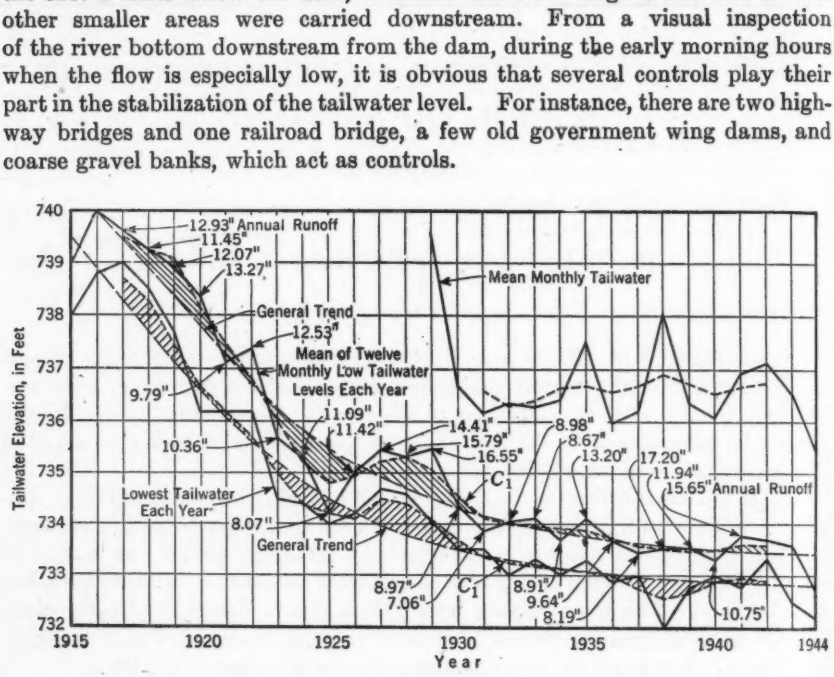

FIG. 8.—TAILWATER DROP AT PRAIRIE DU SAC PLANT ON WISCONSIN RIVER

Mr. Steinmetz' description of the shifting Wisconsin River bottom outlines the conditions very well. Some years ago A. Schoklitsch found, by laboratory experimentation, that the amount of bed load in a stream is proportional to the square of the hydraulic slope and the difference of the actual discharge Q and the discharge Q_o at which the material begins to move. With the change in gradient from 2.5 ft to 1.5 ft per mile below the Prairie du Sac Dam, the carrying capacity in the first seven miles downstream is probably less than one third of that experienced in the early history of the dam.

Before the final design for improving conditions below the Prairie du Sac Dam had been selected, many alternate plans were considered—for example, auxiliary dams for tailwater control in the immediate vicinity downstream, or a combined power plant and dam some distance downstream which would control the minimum tailwater at the Prairie du Sac plant. With the apron added in 1935 and the control weir with crest at El. 734 below the powerhouse, it was found that a maximum amount of benefit could be obtained with a minimum of expenditure. Prior to the improvement, the maximum head permitted was

32 ft, so that as the tailwater dropped the headwater was maintained accordingly. With the improvement made, the stability of the structure was assured, and the average annual output increase was estimated at 15,000,000 kw-hr per yr. For a low water year, this increase was estimated at 22,000,000 kw-hr per yr; and for a high water year only 6,000,000 kw-hr per yr. On the basis of the average yearly increase in output at only 5 mills per kw-hr for the energy, the increase in income for four average years was sufficient to pay the cost of the improvement. In addition to the financial benefits, the stability of the structure is excellent. The step down of the hydraulic gradient through the structure is a valuable improvement.

Observations on the 100-to-1 scale model of the entire dam, lock, and powerhouse were quite convincing in their testimony that the island below the east half of the spillway and the method of gate operation were factors in the development of the 57-ft hole some distance below the lock and dam. It was soon found that all Tainter gates should be opened uniformly across the dam for best results. Much time was spent on the location of the baffle on the new apron, with and without the old baffle, the width of the new apron, and the depth of setting until all these were adjusted so that the velocity at the end of the new apron was as nearly zero as possible; or, if any velocity were present, it was small and in an upstream direction, with the highest velocity on the surface of the water.

Mr. Steinmetz emphasized the economic advisability of model testing. The paper described only a few of the model tests. Many of them were run using types of improvements developed in other places and as suggested by engineers with many years of experience in the design of protective works on dams. Most of these corrective measures proved to be of little or no value. It is much more feasible to arrive at the solution of such a problem by a trial-and-error method in the laboratory than in the field. The adequacy of the hydraulic jump in destroying energy of flowing water was appreciated at the time the model testing started. The early testing consisted of using different methods of causing turbulence on the apron of the dam to destroy energy so that a depth of flow near the bucket of the dam could be attained which would permit the formation of a jump on or near the apron. Blocks attached to the apron to increase roughness and staggered piers were found to be somewhat effective under certain limited conditions of flow. A so-called "cutter pier," with an edge pointed upstream and with curved sides extending downstream and outward shaped like the moldboard of a plow, was placed with its center line downstream from each gate. It was of no value because it accentuated erosion at the end of the apron under certain conditions of flow. Water rose over this pier nearly to headwater elevation and dropped to the river bottom, eroding deep holes at the end of the apron. A continuous wall or weir of proper height, placed a certain distance from the end of the apron, was found to be the most effective protection for the original dam and for the new apron.

Mr. Laushey describes the interesting work done at Carnegie Institute of Technology, Pittsburgh, Pa., including the conditions under which the baffle on the old apron of the Prairie du Sac Dam was effective in producing a hydraulic jump. It is true, as he states, that the production of a jump is of little im-

portance if high velocities reach the bottom. In the model tests, particular attention was given to bottom velocities rather than to the formation and the exact location of the jump.

Mr. Streiff is quite right in stating that erosion below dams is not confined to friable soil foundations, but may occur with any type of foundation. The rapidity of the drop in tailwater depends upon the original natural slope in the river, the kind of material, and the location of the structure relative to the tailwater levels for various flows. A dam on the Niobrara River near Spencer, Nebr., was inspected in 1928, and again in 1934. This structure is on shale foundation, and the original slope of the river was about 7 ft per mile. In this 6-year period the tailwater had dropped 7 ft. The dam on the Wisconsin River at Wisconsin Dells, known as the Kilbourn Dam, is on sandstone rock, but the river bottom downstream is similar to that at the Prairie du Sac Dam. The tailwater at this plant dropped from El. 813 in 1909 to El. 806 in 1935 when modifications were made in this structure.

Mr. Streiff cites two dams in which tailwater erosion occurred on a rock bottom. Model testing is often advantageous even though the site of a dam is on rock and no future lowering of the tailwater elevations is anticipated. Subsequent to the Prairie du Sac Dam tests another model of an existing dam was tested because considerable erosion had occurred in a sandstone bottom of the prototype. This erosion resulted from the plunging of the jet from the apron to the river bottom through a considerable depth of tailwater. This unfavorable condition would have existed had there been no recession of general tailwater elevations. An interesting condition was noticed in the last mentioned test. It was found, under certain conditions of flow, that the jet might either plunge to the bottom, causing high erosive bottom velocities, or it might remain on top. Over a certain range of flow in the model this jet could be caused to plunge or to remain on top by directing it with a shovel or a board. On a rising flood the jet might continue to plunge to the bottom over a "flow-tailwater head" range in which it would remain near the surface on a receding flood. During the period of model testing, which was run at the dam site, a high flood occurred in the river. It was noted that the conditions of the jet as observed on the prototype appeared to be precisely the same as disclosed by the model. The model tests showed that a comparatively narrow apron with a continuous baffle wall would eliminate bottom erosion. There was found to be only one height and one location of baffle which would be effective under all conditions. At this dam, bottom velocities were influenced more by changes in the height and position of the baffle than on any other dam studied by the writers.

Generally, a dam should be designed, at the very beginning, to discharge flood flows without serious tailwater erosion. It is usually much more costly to add protective works at a later date. After serious erosion occurs, the old structure is usually less safe and there is often some hazard involved during the progress of construction of new protective works. Some provision should be made in the original design to take care of conditions should there be some future lowering of the tailwater due to the presence of the dam itself or to other changes such as removal of bridges or straightening of the river channel downstream from the dam.

On the other hand, it may be more economical to construct the original dam on a stream in a safe manner, but not to design it for the ultimate expected drop in tailwater. Then, as necessity requires, model tests may be made and a design made to meet the conditions. In all probability the general plan of procedure used on the Prairie du Sac Dam was the economical way to do this work. The power company could not have spent the additional amount for the extra work in 1914, and to carry the extra investment over a long period of years would have been a burden.

The forty-one Tainter gates, 14 ft by 20 ft (mentioned by Mr. Streiff in item (5)), are capable of passing 160,000 cu ft per sec. The maximum flood since 1873 occurred in 1938 and amounted to 90,000 cu ft per sec. The pond area is about 9,000 acres at a normal water level of 774.0 ft. No difficulties from ice have been experienced. The ice usually deteriorates and softens before passing the gates. There has been no excessive ice pressure, and the pond is permitted to freeze with no provision for relieving the pressure. The turbine units are capable of using about 15,000 cu ft per sec, which is 1.65 cu ft per sec per sq mile on the 9,125 sq miles of drainage area. Since the power load on the company's system is large enough to absorb the output, there is little need to open the gates until a substantial flood reaches this plant. This gives sufficient time for the ice to break up.

It was stated in the paper that small current meters and pitot tubes did not give satisfactory results in the 1929 tests. In subsequent studies of various models of dams the use of pitot tubes has been found very feasible. In fact, the results of the velocity measurements would be relied upon in future tests. A type of pitot tube and its use was developed which avoided most of the troubles encountered in 1929. The pitot tubes were made from 1/8-in. glass tubes using a blow torch to form the bent ends. For approximate and preliminary studies, a rubber bulb attached to both legs of the tube permitted raising the liquid in the glass legs above water level in the channel where the differential head could be observed. Air carried in the flowing water and entering the pitot tube which gave so much trouble in other pitot tubes could readily be detected when it entered the glass tubes. For more accurate determinations a differential fluid gage, using gasoline, was connected to the pitot tube. A method of flushing was developed which permitted successful removal of air from the tubes. Because of great turbulence in most of the model tests and the considerable amount of entrained air, the pitot tubes and differential gage as originally used were not practicable. Clear water under pressure was brought to the gage below the lowest gage fluid level. A valve in each branch hose connecting to the legs could be adjusted so that when the flushing water was turned on the flow in each leg was such as not to cause the gage fluid to be blown out. The procedure was to place the pitot tube in proper position, open the flushing valve, close flushing valve, and read the gage. This was repeated until two or three consecutive readings gave consistent results. The pitot tubes were calibrated by observing the differential head as recorded by the differential gage when the tube openings were held near the surface of flowing water in a channel. The actual velocity of flow was determined by timing the motion of floats.

Mr. Keays took some exception to the writers' use of the term "tailwater erosion." Although tailwater elevations were measured downstream from the turbulent area, the disturbed or turbulent reach was considered to be "tailwater." The general lowering of the bed of the Struma River in Macedonia appears quite similar to that of the lower Wisconsin River. It is possible that flood flows of 500 cu m (17,700 cu ft) per sec or greater may not have created any greater erosion at the dam on the Struma River than existed at lower flows. Maximum floods at Prairie du Sac did not cause maximum erosion. Floods of about one-half maximum resulted in deeper cutting of the river bottom near the end of the apron than greater floods.

The idea submitted by Mr. Keays to develop a roll in the flow downstream from the apron does not seem desirable because of the varying size of floods that may be encountered. However, the design might work perfectly under certain conditions of flood and tailwater level. The Spencer Dam on the Niobrara River was designed so that the water formed a roll which ultimately cut back under the dam as the tailwater receded, and the dam finally failed.

Mr. Steinmetz compared bottom velocities in runs No. 27 and No. 26 and emphasized the need for building up the tailwater to normal for a given flood. Operation on this basis is vital where erosion of appreciable magnitude has occurred. All gates should be opened progressively, allowing flood flows to discharge as uniformly over the entire length of the dam as is practicable at all times. If only one gate at Prairie du Sac were opened to discharge 2,000 cu ft per sec it would be discharging into a tailwater at an elevation of about 735 ft. Since the rate for one gate corresponds to a flood flow of about 80,000 cu ft per sec for the entire dam, the single gate would be discharging into a tailwater which is about 15 ft too low. Erosion increases quickly with tailwater lowered below normal. When using models with sand bottoms, care must be taken to prevent the lowering of tailwater below the desired limit to be tested. Tailwaters must be adjusted downward in the tests.

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DISCUSSIONS

STRENGTH OF BEAMS AS DETERMINED BY LATERAL BUCKLING

Discussion

BY OLIVER G. JULIAN, AND KARL DE VRIES

OLIVER G. JULIAN,[45] M. ASCE.—The formulas given in this paper are the acme of simplicity and far more rational than most specification requirements. With the notable exceptions of members—(*a*) which are unflanged, asymmetrical, or have boxed cross sections; (*b*) those for which $\sqrt{1 - I_y/I_x}$ is appreciably different from unity; (*c*) those with limber webs such that cross sections may distort in their planes so that the effective torsional rigidity of the member is not wholly effective; and (*d*) those with thin wide flanges such as are made from sheet metal—the author's formulas should give results which are fairly reasonable and conservative. It should be especially emphasized that, as plainly stated[46] by Professor Timoshenko, the Timoshenko formulas (with which Mr. de Vries compares his formulas) were derived for deep narrow members such as I-beams rather than wide-flanged members and do not take account of relative horizontal to vertical stiffness.

Notation.—In addition to the notation introduced by the author, the following listed letter symbols are used in this discussion:

$a^2_1 = \dfrac{E I_y}{G K}\left(\dfrac{h}{2}\right)^2$;

G = modulus of rigidity;

H = distance of the load above the shear center; if the load is below the shear center, H is negative;

h = distance between flange centroids;

j = ratio of the product of l and the critical transverse loading force to the critical moment; for pure flexure j equals unity;

NOTE.—This paper by Karl de Vries was published in September, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1946, by George Winter, and David B. Hall; February, 1947, by Theodore R. Higgins; March, 1947, by Neil Van Eenam; April, 1947, by H. N. Hill, and H. D. Hussey; May, 1947, by H. G. Brameld; and June, 1947, by Edwin H. Gaylord.

[45] Chf. Structural Engr., Jackson and Moreland, Cons. Engrs., Boston, Mass.

[46] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, p. 239 et seq.; also see section headings 49 et seq.

m = a factor;
n = a factor;
P_{cy} = Euler's critical load for failure of a pin-ended column by flexure about a principal axis y;
P_w = Wagner's critical load for failure of a column by torsion; thus —

$$P_w = \frac{1}{J^2}\left[G\,K + P_{cy}\left(\frac{h}{2}\right)^2\right] = \frac{G\,K}{J^2}\left[1 + \pi^2\left(\frac{a_1}{l}\right)^2\right] \ldots\ldots (108)$$

r_y = radius of gyration about a principal axis y;
S = section modulus about the x-axis; and
J = polar moment of inertia.

The earliest work on the elastic stability of beams with which the writer is acquainted is that of A. G. M. Michell,[47] published in 1899. Based on geometrical relations, on the differential equations of equilibrium, and on the assumption that the flexural rigidity about the y-axis may be of the same order as, or much smaller than, that about the x-axis whereas the torsional rigidity about the longitudinal z-axis is small as compared to the flexural rigidity about the x-axis, Mr. Michell derives expressions defining the critical loads for symmetrical prismatic and cylindrical beams, subjected to various types of loads including the case in which opposite forces are applied parallel to the axis at symmetrical points in the plane of the greatest rigidity. Rearranged, Mr. Michell's expression defining the first mode of instability for the latter case is

$$M^2 = \left(\frac{\pi^2}{l^2} - \frac{P}{E\,I_y}\right)\frac{E\,I_y\,G\,K}{1 - I_y/I_x} \ldots\ldots (109)$$

in which P and M represent the concomitant critical axial load and moment, respectively. If I_y/I_x is neglected, Eq. 109 is identical with the result[48] given by Professor Timoshenko for a similar case in which I_y is considered as of a smaller order than I_x. For pure flexure and in terms of critical fiber stress, Eq. 109 becomes

$$f^2 = \frac{\pi^2\,d^2\,E\,I_y\,G\,K}{4\,l^2\,I^2_x\,(1 - I_y/I_x)} \ldots\ldots (110)$$

Neglecting I_y/I_x and using the following rough approximations—$\pi^2 = 10$; $I_x = \frac{d^2}{2}\,b\,t$; $I_y = \frac{b^3\,t}{6}$; $K = \frac{2}{3}\,b\,t^3$; and $E\,G = 360 \times 10^{12}$—Eq. 110 yields the author's basic Eq. 7.

The rationalization of the author's basic equation from the Michell theory involves:

1. Considering the member as subjected to equal opposed end couples rather than to transverse loads.

[47] "Elastic Stability of Long Beams Under Transverse Forces," by A. G. M. Michell, *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*, Vol. XLVIII, 5th Series, 1899, p. 298.

[48] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, p. 243, Eq. (*h*).

2. Neglecting the flexural rigidity of the flanges about the y-axis of the beam, which amounts to equating P_w to $\frac{G\,K}{J^2}$ or equating a_1 to zero. For short beams this may lead to ultraconservative design.

3. Neglecting the torsional rigidity of fillets and the web, and using the value of K applicable to extremely wide thin rectangles. The first may lead to errors in K of 25% or more on the conservative side, whereas the latter errs on the daring side.[49,50]

4. Neglecting I_y/I_x as compared to unity. For many members in common use, this may lead to errors of 25% or more on the conservative side.

5. Assuming that the beam flanges and web act as a unit without appreciable warping of cross sections in their planes.[20]

6. Assuming that the areas of the flanges are fully effective in resisting flexure. In the case of thin wide flanges this may lead to errors of dangerous magnitude.[51]

7. Assuming that the cross section of the member is symmetrical or has point symmetry, as for a Z-bar.

The sum total of the errors that may be introduced by assumptions 1 to 7 may be far from negligible.

For a flanged beam, the formula corresponding to Michell's solution for a symmetrical prismatic member subjected to end couples is

$$M^2 = \frac{\pi^2 E I_y G K \left[1 + \pi^2 \left(\frac{a_1}{l}\right)^2\right]}{l^2 (1 - I_y/I_x)} = \frac{J^2 P_{cy} P_w}{1 - I_y/I_x} \quad \ldots\ldots (111a)$$

For purposes of comparison it will be found convenient to write Eq. 111a as

$$M^2 = n^2 \frac{\pi^2}{l^2} E I_y G K \left[1 + \pi^2 \left(\frac{a_1}{l}\right)^2\right] = \frac{m^2 E I_y G K}{l^2} \quad \ldots\ldots (111b)$$

in which

$$n^2 = \frac{1}{1 - I_y/I_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (112a)$$

and

$$m^2 = n^2 \pi^2 \left[1 + \pi^2 \left(\frac{a_1}{l}\right)^2\right] \quad \ldots\ldots\ldots\ldots\ldots (112b)$$

A number of values of n and m are shown in Fig. 14(a). If the torsional rigidity of the member is negligible[20] (as may be the case if the web is thin and deep) so that cross sections may deform readily in their planes, Eq. 111a, using the

[49] "Theory of Elasticity," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, p. 248, "Table of Constants for Torsion of a Rectangular Bar."

[50] "Structural Beams in Torsion," by Inge Lyse and Bruce G. Johnston, *Transactions*, ASCE, Vol. 101, 1936, p. 861, Eq. 6.

[20] "Lateral Stability of Unsymmetrical I-Beams and Trusses in Bending," by George Winter, *ibid.*, Vol. 108, 1943, p. 247.

[51] "Strength of Thin Steel Compression Flanges," by George Winter, *ibid.*, Vol. 112, 1947, p. 527.

value of a_1 given in the notation of this discussion, reduces to

$$M^2 = P^2_{cy}\left(\frac{h}{2}\right)^2 \frac{1}{1 - I_y/I_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (113)$$

For a symmetrical, uniformly loaded, simple beam with the load rigidly attached to the member a distance H above the centroidal axis an approximate solution corresponding to Eq. 111*a* is

$$M = 1.13\frac{\pi\sqrt{E\,I_y\,G\,K}}{l\,(1 - I_y/I_x)}\left\{\sqrt{\left[1 + \pi^2\left(\frac{a_1}{l}\right)^2\right]\left(1 - \frac{I_y}{I_x}\right) + 2.07\left(\frac{a_1}{l}\right)^2\frac{H^2}{(h/2)^2}} - 1.43\,\frac{a_1}{l}\,\frac{H}{h/2}\right\} \ldots (114)$$

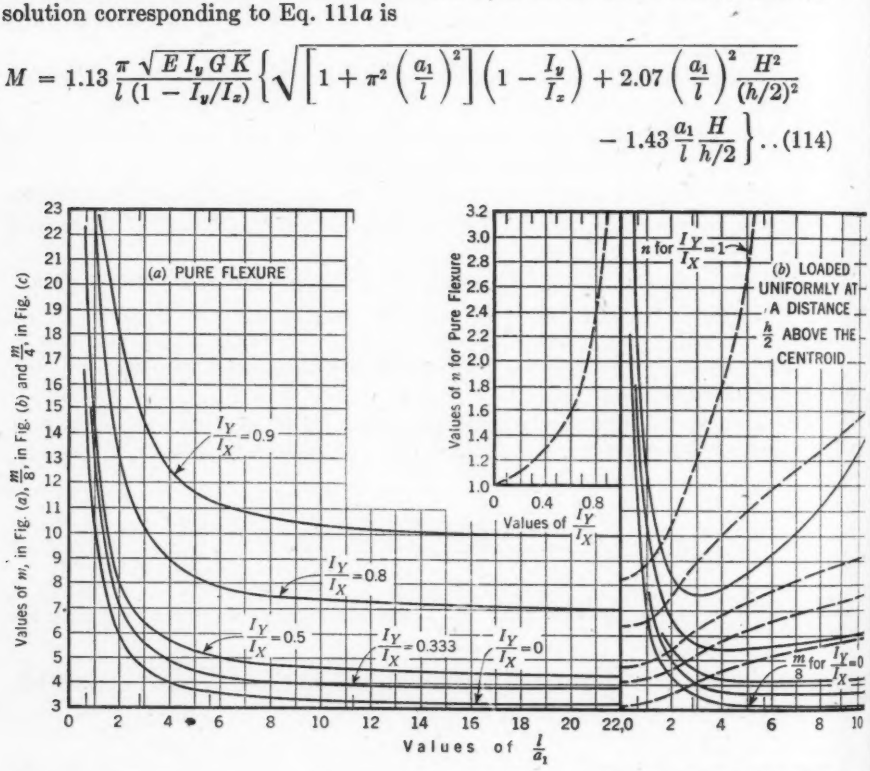


FIG. 14.—VALUES OF n AND m FOR A

This solution is derived by considering the exchange of energy that occurs while the beam is twisting and buckling, account being taken of the deformation about the x-axis as well as that about the y-axis and the z-axis. Eq. 114 can be written

$$M^2 = \frac{n^2\,\pi^2}{l^2}\,E\,I_y\,G\,K\left[1 + \pi^2\left(\frac{a_1}{l}\right)^2\right] = \left(\frac{m}{8\,l}\right)^2 E\,I_y\,G\,K \ldots . (115)$$

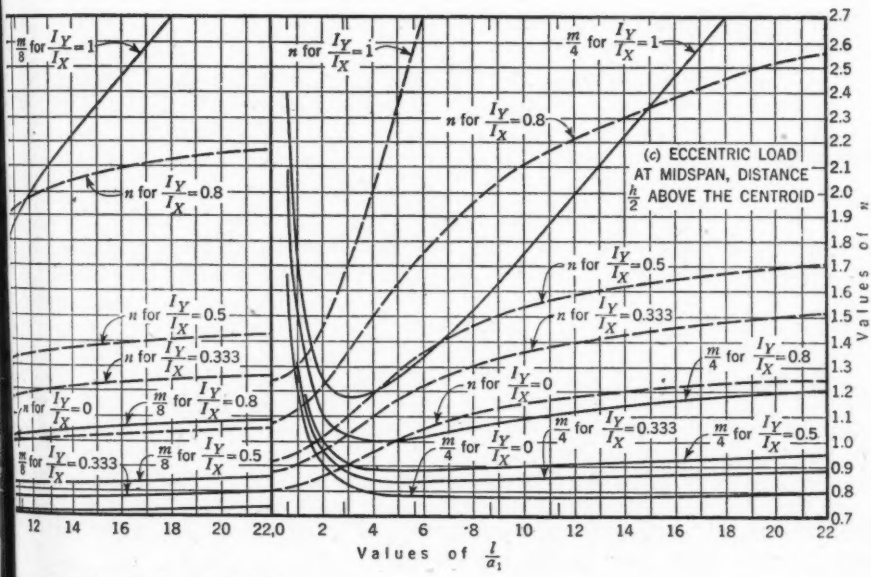
in which

$$n = \frac{1.13}{1 - I_y/I_x}\left[\sqrt{1 - \frac{I_y}{I_x} + \frac{2.07\,\frac{H^2}{(h/2)^2}}{\pi^2 + (l/a_1)^2}} - \frac{1.43\,\frac{H}{h/2}}{\sqrt{\pi^2 + (l/a_1)^2}}\right] \ldots (116a)$$

and

$$m = \frac{28.4}{1 - I_y/I_x}\left\{\sqrt{\left[1 + \pi^2\left(\frac{a_1}{l}\right)^2\right]\left(1 - \frac{I_y}{I_x}\right) + 2.07\left(\frac{a_1}{l}\right)^2 \frac{H^2}{(h/2)^2}} - 1.43\,\frac{a_1 H}{l\,h/2}\right\} \quad ..(116b)$$

A number of values of n and $\frac{m}{8}$ are shown in Fig. 14(*b*). From a comparison of these values and those pertaining to pure flexure, it will be noted that, when $\frac{l}{a_1}$ is small, the critical moment for this case is considerably less than that for a similar beam subjected to pure flexure. If I_y/I_x is small, the beams are



Simple Symmetrical Flanged Beam

about on a par when $l/a_1 = 11.3$; this value of l/a_1 increases as I_y/I_x increases. When l/a_1 is large, the critical moment for the uniformly loaded beam is somewhat more than the corresponding moment for a similar beam subjected to pure flexure. Equating h to d Eq. 115 yields the following approximate expression for critical fiber stress,

$$f = \frac{I_y\,d^2\,E}{I_x\,l^2}\,\frac{n\,\pi}{4}\sqrt{\pi^2 + \left(\frac{l}{a_1}\right)^2} \quad .....(117)$$

Eq. 117 is identical to Eq. 4 if the insignificant terms (those beyond the first) in the Fourier series used by the author are neglected and I_y/I_x is equated to

zero. As K approaches zero, the value of M given by Eq. 115 approaches

$$M = \frac{0.565\, P_{cy}\, h}{1 - I_y/I_x}\left[\sqrt{1 - \frac{I_y}{I_x} + 0.21\,\frac{H^2}{(h/2)^2}} - 0.46\,\frac{H}{h/2}\right] \ldots\ldots (118)$$

For a symmetrical simple beam subjected to a concentrated load at midspan, an approximate solution corresponding to Eq. 114 is

$$M = 1.35\,\frac{\pi\sqrt{E\,I_y\,G\,K}}{l\,(1 - I_y/I_x)}\left\{\sqrt{\left[1 + \pi^2\left(\frac{a_1}{l}\right)^2\right]\left(1 - \frac{I_y}{I_x}\right) + 3.03\left(\frac{a_1}{l}\right)^2\frac{H^2}{(h/2)^2}} - 1.74\,\frac{a_1}{l}\,\frac{H}{h/2}\right\} \ldots (119)$$

which can be written

$$M^2 = \frac{n^2\,\pi^2}{l^2}\,E\,I_y\,G\,K\left[1 + \pi^2\left(\frac{a_1}{l}\right)^2\right] = \left(\frac{m}{4\,l}\right)^2 E\,I_y\,G\,K \ldots\ldots (120)$$

in which

$$n = \frac{1.35}{1 - I_y/I_x}\left[\sqrt{1 - \frac{I_y}{I_x} + \frac{3.03\,\dfrac{H^2}{(h/2)^2}}{\pi^2 + (l/a_1)^2}} - \frac{1.74\,\dfrac{H}{h/2}}{\sqrt{\pi^2 + (l/a_1)^2}}\right] \ldots (121a)$$

and

$$m = \frac{16.93}{1 - I_y/I_x}\left\{\sqrt{\left[1 + \pi^2\left(\frac{a_1}{l}\right)^2\right]\left(1 - \frac{I_y}{I_x}\right) + 3.03\left(\frac{a_1}{l}\right)^2\frac{H^2}{(h/2)^2}} - 1.74\,\frac{a_1}{l}\,\frac{H}{h/2}\right\} \ldots (121b)$$

A number of values of n and $m/4$ are shown in Fig. 14(*c*). As K approaches zero, the value of M given by Eq. 119 approaches

$$M = \frac{0.675\, P_{cy}\, h}{1 - I_y/I_x}\left[\sqrt{1 - \frac{I_y}{I_x} + 0.31\,\frac{H^2}{(h/2)^2}} - 0.56\,\frac{H}{h/2}\right] \ldots\ldots (122)$$

Flexural members are often loaded so that the moment between points of restraint, as regards rotation about their longitudinal axes, is substantially constant. For this case, if the loading is uniformly distributed, a safe but reasonable approximation of the critical moment can be obtained by using Eq. 114 with unity substituted for the constant 1.13. If the loading is concentrated midway between points of torsional restraint, Eq. 119 applies approximately provided 1.00 is substituted for the constant 1.35. If the loading is concentrated at points of torsional restraint, the formulas applicable to pure flexure apply without modification.

The author does not mention cantilever beams. As far as the writer knows, a satisfactory solution of the problem pertaining to a flanged cantilever in which the effective torsional constant is not negligible has not been published to date. Safe, but rough, approximations of the critical moment pertaining to such cantilevers subjected to concentrated terminal loads, can be obtained from Eq. 119, for simple beams, provided 1.27 is substituted for the constant 1.35.

TABLE 9.—EFFECT OF TYPES OF LOADING AND END CONDITIONS

Case	Type of loading	End conditions	n
1	Pure bending	....	1.00
2	Concentrated load at midspan	Fixed ends[a,b]	1.06
3	Uniformly distributed load	Simply supported	1.13
	Concentrated Load at:		
4	The free end	Cantilever	1.27
5	Midspan	Simply supported	1.35
6	0.35 times span length	Simply supported	1.38
7	0.15 times span length	Simply supported	1.53
8	Pure bending	Fixed ends[b]	2.00
9	Uniformly distributed load	Cantilever	2.04

[a] Ends fixed against rotation about the x-axis. [b] Ends fixed against rotation about the y-axis.

Various cases in which the load is placed at a considerable distance above or below the centroidal axis have been analyzed[52] by H. S. Richmond by investigating the differential equations of equilibrium.

The marked effect of various types of loading and end conditions on the value of the critical moment may be further appraised by a glance at Table 9, which pertains to deep narrow prismatic members. The values of n are those involved in the formula:

$$M = \frac{n\,\pi}{l}\,\sqrt{E\,I_y\,G\,K} \qquad (123)$$

The supported ends of all members are considered fixed as regards rotation about a longitudinal axis but, unless otherwise stated, are free to rotate about the x-axis and the y-axis. In all cases, the loading is on a level with the centroidal axis. Table 9 was compiled from data given by John Prescott[53] and by Professor Timoshenko.[2]

It appears unfortunate that the author, in his notation, designated l as "the laterally[53a] unsupported length of the compression flange." A better designation for l would appear to be "length of member between points of fixity as regards rotation about a longitudinal axis." This usage has been adopted in this discussion It should be emphasized that the author's equations, those given in the references, and in this discussion are based on the ends of the member being fixed as regards torque about a longitudinal axis. Ample provision for such restraint is most important but is often neglected in practice.

The columns of free-standing, two-dimensional, frames cannot be said to fix the ends of the beams which are component parts of the frame; the columns will flex normal to the plane of the frame and will thereby allow the ends of the

[52] "Elastic Equilibrium in the Theory of Structures," by H. S. Richmond, *Transactions*, ASCE, Vol. 94, 1930, p. 845.

[53] "Applied Elasticity," by John Prescott, Longmans, Green & Co., London, 1924, p. 499.

[2] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, p. 239, Chapter V.

[53a] Correction for *Transactions:* In September, 1946, *Proceedings*, p. 1011, line 6, change "unsupported length * * *" to "the laterally unsupported length * * *."

beams to rotate about longitudinal axes. For such cases the stability of the frame as a unit should be considered.

The analysis under discussion assumes that the loading is perfectly free to rotate and move transversely. Quite often this is not so in practice (or tests); frequently the loading is so attached to the member as to stabilize it as regards torque and transverse movement.

The writer has grave doubts regarding the safety of the author's proposal (last paragraph under "Conclusions") to allow the same unit stresses in asymmetrical sections as are used for symmetrical sections; nor can he agree that asymmetrical sections are seldom used as long beams. Towers are often built entirely of angles, all of which are subject to flexure due to eccentric end conditions. It is not clear how the author's formulas, or other formulas based on the torsional rigidity of symmetrical sections, can be applied rationally to such a section as an angle or a T-section. If the horizontal leg (flange) is in compression, b and t may be taken from its cross-sectional dimensions, but it cannot be said that the compression flange is supported by the tension flange; the latter is not present. Also, it is evident that an angle or tee has considerably less torsional rigidity than an I-beam or H-section, the flanges of which have values of b and t identical with those of the angle or tee. H. N. Hill,[54] Assoc. M. ASCE, and George Winter,[20] M. ASCE, have given formulas applicable to asymmetrical I-beams of such proportions that $\sqrt{1 - I_y/I_x}$ is approximately equal to unity. J. N. Goodier has given[55] formulas applicable to deep asymmetrical members subjected to axial load as well as flexure.

Many members, such as crane girders, are ordinarily subjected to flexure about both principal axes, together with torque about a longitudinal axis. This case is not covered by the author. C. O. Dohrenwend, Assoc. M. ASCE, has analyzed[56] this problem for the special case in which a prismatic member is subjected to pure flexure about both principal axes, together with an applied torque at midspan, I_y and K being assumed small as compared to I_x.

The author's method of analyzing the stability of a plate girder appears to contain a series of judgment adjustments for which he no doubt has ample empirical justification. It is hoped that, in his closing discussion, he will explain fully his reasons for these adjustments.

Although included in the scope of the title, the author's equations are obviously not meant to apply to light-gage structural members made of sheet metal of such dimensions that the entire section is not fully effective in resisting torsion and flexure. Such members (made of steel) have been analyzed by Professor Winter using semi-empirical methods.[51] The results of this analysis have been incorporated, in abbreviated form, in specifications of the American Iron and Steel Institute (A.I.S.I.)[25] issued in April, 1946.

Flexural members are often subjected to a combination of at least three types of loading—end couples, distributed loads, and concentrated loads. As

[54] "The Lateral Instability of Unsymmetrical I-Beams," by H. N. Hill, *Journal of the Aeronautical Sciences*, Vol. 9, No. 5, 1942.

[55] "Flexural-Torsional Buckling of Bars of Open Section," by J. N. Goodier, *Bulletin No. 28*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., 1942.

[56] "Action of Deep Beams Under Combined Vertical, Lateral and Torsional Loads," by C. O. Dohrenwend, *Journal of Applied Mechanics*, September, 1941, p. A-130.

[25] "Specifications for the Design of Light Gage Structural Members," A.I.S.I., April, 1946.

has been indicated, the critical moments for each of these loadings may differ considerably. It can be shown,[53,57] that a sufficient (as distinguished from a necessary) condition of stability is defined by

$$\sum \frac{M}{M_{cr}} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (124)$$

in which the terms in the numerators pertain to the actual loading, whereas those in the denominators pertain to the critical condition for that type of loading acting alone. This simple rule will give results that are ordinarily reasonable although somewhat on the safe side.

The author, quite rightly, introduces transition curves between the proportional limit and the yield-point strength of the material. The validity of these transition curves might be supported by assuming initial imperfections, as has been done by Professor Winter in deriving secant formulas for beams,[19] or by taking the moduli E and G as variables above the proportional limit, and building up a mathematical argument on these premises.

For conventional, symmetrical steel members, subjected to flexure about the principal x-axis, the writer suggests the use of the following in lieu of the author's basic Eq. 7:

$$f = \frac{56.1 \times 10^6}{S\,l}\, n \sqrt{I_y\, K + \frac{6.5\, I^2{}_y\, h^2}{l^2}} = \frac{17.86 \times 10^6}{S\,l}\, \frac{m}{j} \sqrt{I_y\, K} \ldots (125)$$

in which E and G have been taken as 29×10^6 and 11×10^6, respectively. Eq. 125 can be applied to other materials such as the structural aluminum alloys by suitable modifications of these moduli. The most important values of n and m/j encountered in practice might well be tabulated and exhibited graphically in standard handbooks, together with values of K, a_1, $I_y\, K$, and $I^2{}_y\, h^2\ 1/s\sqrt{I_y\, R}$ for each section as has been done for K in the Structural Aluminum Handbook.[58]

By using the approximate values of I_x, I_y, and K given after Eq. 110 and equating h to d, Eq. 125 can be written in the roughly approximate form:

$$f = \frac{18.7 \times 10^6}{l\, d/b\, t}\, n \sqrt{1 + 1.62\, \frac{b^2\, d^2}{t^2\, l^2}} \ldots\ldots\ldots\ldots\ldots (126)$$

If n is taken as unity (its value for pure flexure when I_y/I_x is negligible) and if the second term in the radical is neglected, Eq. 126 is substantially the author's basic Eq. 7 (the difference occurs on account of differences in assumed values of the elastic moduli). However, unity, in many cases, is a poor approximation for n and the second term in the radical may be far from negligible; in fact, for the shorter members, this term is the predominating term.

[57] "Rayleigh's Principle," by G. Temple and W. G. Bickley, Oxford Univ. Press, England, 1933, pp. 31 and 41.

[19] "Strength of Slender Beams," by George Winter, *Transactions*, ASCE, Vol. 109, 1944, p. 1321.

[58] "Structural Aluminum Handbook," Aluminum Co. of America, New Kensington, Pa., 1945.

This accounts for the increasing divergence between the author's formula and others as l decreases.

In Fig. 15 a comparison is made, for a 14-in., 142-lb wide-flanged beam, between the author's basic Eq. 7 and Eq. 125, for two cases frequently encountered in practice—(*a*) pure flexure; and (*b*) a uniformly distributed load on

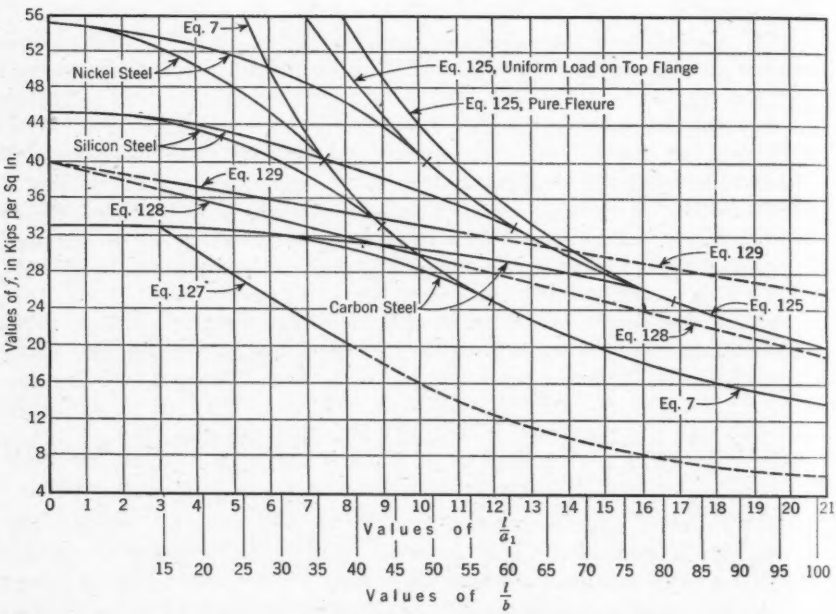


FIG. 15.—COMPARISON OF FORMULAS FOR A 14WF142 BEAM

the top flange, $H = d/2$. Arbitrary transition curves between the approximate proportional limits and the yield points for each of the classes of steel mentioned by the author have been drawn in this figure. For purposes of ready comparison, the loci of the following formulas have also been shown:

$$f = \frac{37{,}200}{1 + \frac{(l/b)^2}{1{,}800}} \quad \ldots\ldots (127)$$

$$f = 40{,}000 - 60\,\frac{l}{r_y} \approx 40{,}000 - 208\,\frac{l}{b} \quad \ldots\ldots (128)$$

and

$$f = 40{,}000 - 40\,\frac{l}{r_y} \approx 40{,}000 - 139\,\frac{l}{b} \quad \ldots\ldots (129)$$

Eq. 127 will be recognized as the A.I.S.C. specification (1936) formula[59] multi-

[59] "Specifications for the Design Fabrication and Erection of Structural Steel for Buildings," A.I.S.C., 1936.

plied by 1.65, which is the safety factor implied therein. Eqs. 128 and 129 between $\frac{l}{r_y} = 50$ and 190 $\left(\frac{l}{b}$ between 15 and 55 approximately$\right)$ represent the results of four sets of tests on I-beams and H-beams reported[60] by Herbert F. Moore in 1913. Eq. 128 applies to pure flexure, whereas Eq. 129 applies to the uniformly distributed load case (without taking account of the effect of H). The parts of the loci of these equations which are beyond the somewhat limited range of the tests are shown dotted in Fig. 15. These test results appear to indicate clearly that transition curves such as those introduced by the author are fully warranted.

All the beam formulas given in this discussion pertain to critical (ultimate) conditions. In practice, these formulas should be divided by an appropriate safety factor; or, preferably, the loads should be multiplied by a suitable factor of overload. The latter procedure is especially preferable if the member under consideration is subjected to a combination of flexural and axial loads, in which case the principle of superposition obviously does not apply.

Mr. de Vries is to be commended for having performed a series of tedious calculations most precisely and for again raising an important, but somewhat controversial, subject for discussion. It is unfortunate that test results over an extended range of member sizes, shapes, lengths, loading, and end conditions, with which the author's results and other calculations may be compared, apparently are not available. The writer regrets that space limitations do not permit discussing the paper in detail or to the extent it fully warrants.

In addition to the references cited by the author, and in discussion, several others are especially worthy of note.[61,62,63,64,65,66,67] No attempt has been made to make this list complete.

The writer is indebted to James Adam, Jr., for aid in preparing the illustrations in this discussion.

KARL DE VRIES,[68] M. ASCE.—The general tenor of the discussions has been to the effect that the writer's proposed working, or specification, formula, Eq. 9, is reasonable for most conditions, but that it falls down for very short, slender beams and for plate girders. The writer, in this closing discussion, will attempt to show wherein the discussers have made approximations which he

[60] "The Strength of I-Beams in Flexure," by Herbert F. Moore, *Bulletin No. 68*, Univ. of Illinois, Urbana, 1913.

[61] "Problems Concerning Elastic Stability in Structures," by S. Timoshenko, *Transactions*, ASCE, Vol. 94, 1930, p. 1000.

[62] "The Torsional Effect of Transverse Bending Loads on Channel Beams," by F. B. Seely, W. F. Putnam, and Wm. L. Schwalbe, *Bulletin No. 211*, Univ. of Illinois, Urbana, 1930.

[63] "Elastic Instability of Members Having Sections Common in Aircraft Construction," by G. W. Trayer and H. W. March, *Report No. 382*, National Advisory Committee for Aeronautics, Washington, D. C., 1931.

[64] "Strength of Light I-Beams," by Milo S. Ketchum and Jasper O. Draffin, *Bulletin No. 241*, Univ. of Illinois, Urbana, 1932.

[65] "The Lateral Stability of Equal-Flanged Aluminum-Alloy I-Beams Subjected to Pure Bending," by C. Dumont and H. N. Hill, *Technical Note No. 770*, National Advisory Committee for Aeronautics, Washington, D. C., 1940.

[66] "Strength of Materials," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., Vol. 2, 2d Ed., 1941, p. 229.

[67] "Lateral Buckling of I-Section Column with Eccentric End Loads in Plane of Web," by Bruce Johnston, *Journal of Applied Mechanics*, December, 1941, p. A-176.

[68] Designer, Bethlehem Steel Co., Fabricated Steel Constr., Bethlehem, Pa.

considers incorrect, or have used assumed loadings or conditions which practically cannot exist. The writer also proposes modification of specification requirements to permit the use of the full basic unit stress, without reduction, for the case of braced beams.

The writer believes firmly that Eq. 4 is the best and simplest formula for beams in buckling; and substitutions of approximate values for its ratio of

TABLE 10.—CRITICAL STRESSES FROM EQ. 4, FAILURE FOR UNIFORM CENTROID LOADS AND FOR

$\frac{l}{a}$	l (ft)	$\frac{l}{b}$	$\frac{l\,d}{b\,t}$	Load at Centroid: f_{cc} (Eq. 4)	Load at Centroid: $24 \times 10^6 \div \frac{l\,d}{b\,t}$	Load at Centroid: $\frac{\text{Col. 5}}{\text{Col. 6}}$	Load at Top: f_{ct} (Eq. 4)	Load at Top: $20 \times 10^6 \div \frac{l\,d}{b\,t}$ (Eq. 7)
				Pounds per Square Inch			Pounds per Square Inch	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
				36WF194				
4	36.6	36.3	1,050	27,400	22,860	1.20	20,730	19,050
5	45.8	45.4	1,313	20,360	18,280	1.11	16,000	15,230
6	55.0	54.4	1,576	16,210	15,230	1.06	13,130	12,690
7	64.1	63.5	1,838	13,490	13,060	1.03	11,200	10,880
8	73.3	72.6	2,101	11,560	11,420	1.01	9,790	9,520
9	82.4	81.6	2,363	10,130	10,160	1.00	8.720	8,460
10	91.6	90.7	2,626	9,020	9,140	0.99	7,870	7,620
11	100.7	99.8	2,889	8,130	8,310	0.98	7,180	6,920
				36WF160				
3	32.2	32.2	1,135	28,500	21,150	1.35	20,580	17,620
4	42.9	42.9	1,513	18,760	15,860	1.18	14,200	13,220
5	53.6	53.6	1,891	13,940	12,690	1.10	10,950	10,580
6	64.3	64.3	2,269	11,100	10,580	1.05	8,990	8,810
7	75.0	75.0	2,647	9,240	9,070	1.02	7,670	7,560
8	85.7	85.7	3,026	7,920	7,930	1.00	6,710	6,610
9	96.4	96.4	3,404	6,940	7,050	0.98	5,970	5,880

moments of inertia and simplifications of its k-values, listed in Table 1, will lead to approximated or distorted results, which should not be compared critically with the true values determined from Eq. 4. Formulas for beams in pure bending are generally used by the discussers. They are simpler than those for most other loadings, but, if other formulas are available, they do not deserve this consideration, as will be proved in this discussion.

The simple beam, with its own weight uniformly distributed along its centroid, will have its lowest critical stress if its carried load is uniformly distributed along the top flange (see k-values in Table 1). For relatively short beams the weight of the beam will be a small part of the total load; but as span lengths increase, the weight of the beam will form an increasing portion of the total load until the beam is capable of supporting only its own weight. Uniform top

flange and centroid loads, then, should be the ones considered for writing a specification formula of the type of Eq. 9. Mr. Van Eenam finds disagreement on centroid load stresses for two 36WF beams, but his computations are based on approximate values for I_x, I_y, and k. Since the true values are known, the critical stresses for centroid loading have been computed accurately by the writer and are given in Table 10. The values thus determined show that,

FORMULAS FROM TABLE 4, AND SAFETY FACTORS FROM EQ. 130
UNIFORM TOP FLANGE LOADS, ON A SIMPLE BEAM

Flange								
Col. 8 / Col. 9	$12 \times 10^6 + \frac{l\,d}{b\,t}$	$\frac{f_c}{\frac{12\,w\,l^2}{8} \times \frac{d}{2\,I_x}}$	Col. 11 minus Col. 12 (f_t)	$\frac{f_c}{f_{cc}}$ Col. 12 / Col. 5	$\frac{f_t}{f_{ct}}$ Col. 13 / Col. 8	Col. 14 plus Col. 15	$\frac{n}{1}$ Col. 16 (Eq. 130)	$\frac{l}{a}$
	Pounds per Square Inch							
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(1)
			36WF194					
1.09	11,429	588	10,841	0.021	0.523	0.544	1.84	4
1.05	9,139	920	8,219	0.045	0.514	0.559	1.79	5
1.03	7,614	1,324	6,290	0.082	0.479	0.561	1.78	6
1.03	6,529	1,802	4,727	0.134	0.422	0.556	1.80	7
1.03	5,712	2,354	3,358	0.204	0.343	0.547	1.83	8
1.03	5,078	2,979	2,099	0.294	0.241	0.535	1.87	9
1.03	4,570	3,678	892	0.408	0.113	0.521	1.92	10
1.04	4,154	4,451	..	..	..	..	..	11
			36WF160					
1.17	10,573	459	10,114	0.016	0.491	0.507	1.97	3
1.07	7,931	815	7,116	0.043	0.501	0.544	1.84	4
1.03	6,346	1,274	5,072	0.091	0.463	0.554	1.81	5
1.02	5,289	1,834	3,455	0.165	0.384	0.549	1.82	6
1.02	4,533	2,497	2,036	0.270	0.265	0.535	1.87	7
1.02	3,966	3,261	705	0.412	0.105	0.517	1.93	8
1.02	3,525	4,127	..	..	..	..	..	9

for these two beams, consistent safety factors for the most unfavorable loading exist. The 36WF194 and 36WF160 sections have the torsional bending constants $a = 109.9$ in. and $a = 128.6$ in. respectively.[69] Their critical stresses for uniform centroid and top flange loads are computed in Cols. 5 and 8, Table 10; the corresponding failure stresses computed from the formulas of Table 4 are given in Cols. 6 and 9; and the ratios of critical stress to the corresponding failure stress are shown in Cols. 7 and 10. Col. 11, Table 10, gives the working stress from the proposed specification formula Eq. 9; Col. 12 shows the unit stress resulting from the weight of the beam, w, which is 194 and 160 lb per lin ft, respectively; Col. 13, Table 10, lists the unit stresses that are available

[69] "Bethlehem Manual of Steel Construction," *Catalogue S-47*, Bethlehem Steel Co., Bethlehem, Pa., 1934, p. 285.

for top flange load. The load at the centroid causes the unit stress f_c (Col. 12), but f_{cc} (Col. 5) would be critical; the load at the top flange causes the unit stress f_t (Col. 13) and f_{ct} (Col. 8) would be critical; the critical stress of the beam is reached when both stresses, f_c and f_t, are increased n times, or when

$$\left(\frac{f_c}{f_{cc}} + \frac{f_t}{f_{ct}}\right) n = 1 \qquad (130)$$

in which n is the least factor of safety which the working formula, Eq. 9, has with the critical stress, Eq. 4; n is computed in Cols. 14 to 17. Col. 10, Table 10, gives the close relationship between Eq. 7 and Eq. 4, and Col. 17 shows the nearly constant safety factor regardless of l/a. Since all rolled beams show critical stresses well above the proportional limit for $l/a = 2$ (see also Table 2), similar results will be obtained for all rolled beams, and only special deep sections with thin flanges, having useful lengths below the limit $l/a = 2$, show excessively high safety factors.

To apply Eq. 4 to a thin-flanged deep section with computable a, it was assumed that a rolled beam was cut in the middle of the web, and that 2-ft, 4-ft, and 6-ft additional webs were inserted and welded between the beam halves. It was found for these thin-flanged deep beams that the l/a-ratio for long lengths is larger than 2 and that therefore good agreement of Eqs. 4 and 7 is maintained; but that, for short lengths and increasing depth, Eq. 4 gives critical stresses well above those indicated by Eq. 7. This fault of Eq. 7 has been emphasized by several discussers, especially by Mr. Brameld; but, for reasons explained hereinafter, the writer proposes that the full basic unit working stress be used for such cases.

The treatment of riveted plate girders in the paper has been criticized and preference has been expressed for adherence to Mr. Madsen's[11] findings. Mr. Madsen tested two riveted girders of very small size and thin flanges in torsion and recommended that a K-value equal to one third of that of the identical solid section be used; the writer proposes the modification of Eq. 11, which expresses the properties of the solid section, to Eq. 12. The use of Eq. 12 for thin-flanged girders has the same effect as Mr. Madsen proposes, but the reduction for heavy-flanged girders is less by Eq. 12, and should be less. The writer believes this reduction is necessary not so much for rivet slip, but mostly for the overestimated contribution of the vertical flanges to the K-value of the solid section. In the example of the laminated section of Fig. 6 and its accompanying computations of K, and not considering the effect of the inscribed circle (46.9 in.^{4}), the vertical flange (54.7 in.^{4}) adds about 16% to the total value of K, and will add a similar percentage to the solid K. In a similar example, if the same flange angles are used and the thickness of the cover plates is reduced, the contribution of the horizontal flange (234.0 in.^{4}) is reduced as the cube of its thickness, whereas the vertical flange, which should have a minor influence on torsional stiffness, appears unreduced and has a major influence on the value of K. The same percentage reduction of K for the solid section should not hold for both examples; and it is found that in applying Eqs. 7 and 12 the stresses of the thin-flanged girder correspond to the

[11] "Report of Crane Girder Tests," by I. Madsen, *Iron and Steel Engineer*, November, 1941, Section 4, p. 68.

use of one third, or less, of the K-value of the solid section in Eqs. 4 and 10, and that for heavier flanged girders the used fraction of K of the solid section is more than one third. The heavy-flanged riveted plate girder, then, showed the same full agreement of Eq. 7 with Eq. 4 as rolled beams and heavy-flanged deep beams; and thin-flanged riveted plate girders showed the same gap between Eqs. 7 and 4 as the corresponding thin-flanged deep beams.

Rolled sections, deep beams, and plate girders with intermediate lateral supports between vertical supports, generally, can be divided into two types. If they are of the deck type, the main load is uniformly distributed along the top flange; since specifications require that the compression flange be stayed by lateral bracing and by cross frames at short intervals, the laterally unsupported lengths are short, and the short beam formulas, Eqs. 8, will limit their working stresses in most cases. If it appears from their properties, however, that they will fall in the range of long beams, then Eq. 7 gives, for them, critical stresses that are too low. This is true for two reasons: First, there is the gap between Eqs. 4 and 7 for short deep beams and plate girders with thin flanges; and, second, at $l/a = 2$, for example, a k-factor of 2.269 for simple beams has been used to compute Eq. 4, whereas, actually, $k = 3.499$ for two-bay beams and is larger for these multiple-bay beams. If they are of the through type with bottom flange bracing and brackets to support the top flange, the laterally unsupported lengths are larger, but limited, since the lengths between vertical supports themselves are limited by specifications; the main load is delivered, say, through floor beams and is a concentrated load at these points; and instead of $k = 2.269$ for $l/a = 2$, as used in Eq. 4 to establish Eq. 7, $k = 5.411$ for two-bay beams and is larger for multiple-bay beams. It seems, then, that no reduction of the basic bending stress is required for beams with intermediate lateral supports, regardless of the unsupported length between the points of attachment of these supports, provided the lengths are within usual specification requirements.

All this adds up to the recommendation that Eqs. 8 and 9 be used for specification purposes if the compression flange of the beam has no intermediate lateral supports between its vertical supports, and that the basic bending stress (for carbon steel, 18,000 lb per sq in. for the American Railway Engineering Association (AREA) and 20,000 lb per sq in. for the American Institute of Steel Construction (AISC)) be used if intermediate lateral supports are provided; t in Eqs. 8 and 9 for plate girders and other compound sections is determined from Eq. 12. It is not proposed to abandon the provision of most specifications that the gross section of the compression flange be not less than the gross section of the tension flange.

Eq. 9 is offered to replace the highly inaccurate l/b-formulas in common use and is conservative for loadings other than the one for which it is derived. For special cases, if specifications do not have to be followed strictly, capacities can be computed from Eq. 4, using the proper k-value from Table 1. The general formulas for k are quite complicated, but Eqs. 21 and 36, which are of most interest, can be simplified by dropping y (see Table 7) and formulas will result which, for simple beams, are correct within a fraction of 1%. For ex-

ample, for uniform top flange load on a simple beam Eq. 36*a* reduces to

$$k = 1.2825 \left(\sqrt{0.481 \frac{l^2}{a^2} + 5.747} - 1 \right) \ldots\ldots\ldots\ldots (131)$$

and for uniform centroid load on a simple beam Eq. 36*b* reduces to

$$k = 0.8895 \sqrt{\frac{l^2}{a^2} + 9.87} \ldots\ldots\ldots\ldots\ldots\ldots (132)$$

and the critical stress may be computed from Eq. 4 since all other terms in that equation are known. A simple method, if a is known as for rolled beams and if l/a is larger than 2, is to increase the working formula, Eq. 9, in the ratio of the proper k-factor of Table 1, Cols. 2, 3, 4, 6, or 7, to that of Col. 5.

The erection problem of picking beams has been mentioned in the paper and Messrs. Hall and Hussey take exception to the statement made. The writer feels that this problem is important enough for a separate paper and beyond the scope of this one.

Professor Winter claims that the writer's formulas do not apply to beams for which $b/t > 20$, and reaches this conclusion from a formula for pure bending. In computing the properties of the beams of Fig. 9, he substitutes actual flange thicknesses in Eq. 7 instead of those determined from Eq. 11, which in this case is preferable to Eq. 12. When t as determined from Eq. 11 is used, close agreement between his and the writer's values results.

Mr. Hall suggests that some wide beams, as the 14WF87 section, fall in the plastic range; the writer's formulas place them there and take full recognition of this fact. Mr. Hall also discusses beams with high $\frac{I_y}{I_x}$-ratios and agrees that it is unnecessary to use his proposed correction factor of Eq. 56. If the correction factor for beams with top flange loading were derived, it would be smaller than the value determined by Eq. 56, and would so complicate Eq. 4 that a general solution of that equation would be impossible. His example of a beam in pure bending, consisting of a single panel of a through plate girder bridge cannot be considered as typical of this condition; it is a multiple-bay beam with load application at the panel points, and gives k-values different from those shown by Mr. Hall.

Mr. Higgins misses in the paper a supporting argument for Eqs. 6. For the steels considered, Col. 5, Table 4, gives proportional limits which are the best the writer could establish, and above which the computed critical stresses do not indicate true stresses. Stresses corresponding to a reduced modulus of elasticity beyond the proportional limit have not been computed; instead, arbitrary transition curves between yield point and proportional limit have been drawn. If the AISC committee on specifications[29] prefers to maintain the basic bending stress up to the intersection with the long beam formula, Eq. 9, then it seems that it introduces a safe and worthwhile simplification, in spite of a slightly lower safety factor which might exist for a short range.

Mr. Hill uses Eq. 75 for centroid load and after making substitutions concludes that the writer's formula for very long lengths may give values 30%

[29] "Specification for the Design, Fabrication and Erection of Structural Steel for Buildings," AISC, New York, N. Y., 1946.

higher than the theoretically correct values. The writer uses Eq. 4 with k of Eq. 36*b* for the same loading and believes it to be more nearly theoretically correct than Mr. Hill's, since the approximations in Eq. 4, considering rolled beams, are entirely negligible. If also, within the practical range of long spans, all beams computed by Eq. 4 check with Eq. 7, then there seems to be no reason why Eq. 7 should give erroneous results. Mr. Hill states that the writer's formulas become increasingly conservative as the span decreases; the writer has already discussed this for the case of top flange loads, and has proved that it leads to a constant safety factor (see Table 10); for other loads this might need consideration especially if high-alloy materials are used.

Mr. Hussey compares the stresses of the proposed formula, Eq. 9, with those of the old AISC specification formula and finds some rolled beams considerably reduced in their allowable stresses; he fails to state that most rolled beams will have their allowable stresses increased. The proposed formula tends to allot to the beam its allowable unit stress according to its computed buckling strength; and if a beam in the past has been designed for its critical stress this practice does not have to be continued if it is found to be faulty. Mr. Hussey prefers the investigation of a single bay of a multiple-bay beam; the writer believes that it is extremely complicated to simulate the loading on such a separate bay and rather considers the multiple-bay beam. Instead of groups of concentrated and partial uniform loads on simple beams, the extreme cases of a concentrated load at the center and uniform load over the full length of the beam are considered in Table 1, Cols. 2 to 7.

Mr. Brameld presents some very unusual cases of laterally braced thin-flanged deep plate girders; the writer believes that Mr. Brameld will agree with him on the proposed treatment of such girders. To compute girder flanges as struts is equivalent to applying the l/b-rule and should be avoided.

Professor Gaylord starts with the pure bending formula and, correcting it for top flange load, derives Eq. 104, which is possibly closer to theory, but more involved and as limited in range as Eq. 9. Professor Gaylord's Eq. 102*a* will apply when $l/a < 2$. It is applicable to thin-flanged deep beams with known or computable torsional bending constant a, but for these rare cases the writer would prefer Eq. 4 with Eqs. 131 and 132, respectively. Examples 1 to 3 indicate conservatism of the proposed formula; Table 10, nevertheless, shows that the apparent understress of Eq. 9 for top flange load is false, and that there is really a nearly constant safety factor for the beam.

Mr. Julian gives a thorough investigation of the problem and proposes, instead of Eq. 7, his Eq. 125, the factors of which, if charted or listed in handbooks for rolled beams, would not be available for other sections. The formula is written for pure bending, and the n-values of Table 9 will not make it usable for other loadings, since n, which should be an equivalent of ratios of k, is not as general as indicated in Table 9, but varies with the length of the beam, as can be seen from Table 1. Eq. 7 is not derived from a formula for pure bending, as is Eq. 109, but is derived as an observed approximation to the plotted results of a formula for top flange loading. The limitations stated by Mr. Julian as

necessary to justify passing from Eq. 109 to Eq. 7 do not exist, because no such method of approach is in question. Eq. 12, as proposed in the paper for compound sections, can also be applied to box sections. In computing cantilever beams, the writer uses twice the cantilever as the laterally unsupported length.

The writer wishes to thank the discussers for the interest they have shown in the subject of the paper; he appreciates the many new ideas they have expressed and which have led the writer to additional investigations and to conclusions incorporated in this discussion.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

RELIEF WELLS FOR DAMS AND LEVEES

Discussion

BY T. A. MIDDLEBROOKS AND WILLIAM H. JERVIS

T. A. MIDDLEBROOKS,[40] ASSOC. M. ASCE, AND WILLIAM H. JERVIS,[41] M. ASCE.—The excellent discussions of this paper have added immensely to the knowledge of relief well design and will be most helpful to engineers who are confronted with this problem.

Mr. Barksdale emphasized the need for careful investigations. In-place permeability pumping tests, mentioned by Mr. Barksdale, are essential for large jobs, where maintenance pumping cost will be excessive. However, in most cases, it is cheaper to install a few additional wells as an added precaution against low discharge estimates than it is to conduct expensive pumping tests. Head loss in the casing and screen, and clogging of wells due to growth, are most important factors, all of which justify larger wells and periodic checks.

Mr. Turnbull's prediction that relief wells will be used on all important engineering structures on pervious foundations is a good possibility as soon as the costs of well installations are lowered by improved techniques. His comment that the foundation may be too pervious for relief wells to remove sufficient water to reduce the pressure to a safe value is a possibility, but a remote one. Collector systems are preferred in most cases; however, they are expensive and should be omitted where it is practicable to do so. The Mississippi River levee installations referred to were experimental and woefully underdesigned. There is no doubt that these installations would have been satisfactory if wells of 6-in. diameter or larger had been used, and if the wells had penetrated deeper into the water-bearing strata. Tight packing around the well is costly and it is not important as long as no provision is made for closing the well during operation.

Mr. Gilboy has emphasized that a wide difference between vertical and horizontal permeability is present in all deposits. This difference is large even

NOTE.—This paper by T. A. Middlebrooks and William H. Jervis was published in June, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1946, by Henry C. Barksdale, Willard J. Turnbull, and Glennon Gilboy; December, 1946, by W. A. Wall and C. A. Stone; January, 1947, by John R. Charles, Horace A. Johnson, P. C. Rutledge, H. H. Roberts and Carter V. Johnson, Frank E. Fahlquist, and Harry R. Cedergren; February, 1947, by John S. McNown; May, 1947, by Reginald A. Barron, Preston T. Bennett, Charles I. Mansur, and S. J. Johnson; and June, 1947, by Kenneth S. Lane.

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[41] Cons. Engr., O. J. Porter & Co., Sacramento, Calif.

in seemingly uniform deposits of sand. His comments on controlled versus uncontrolled seepage are important. A boil serves the same purpose as a relief well; but it is uncontrolled and dangerous.

Fully penetrating wells, as shown by Messrs. Wall and Stone, are definitely preferred. Wall spacing no greater than the distance to the source, as they propose, appears reasonable and installation of wells greater than the theoretical is certainly desirable.

The adoption of grouted outer seals, as proposed by Mr. Charles, is definitely preferred to the use of bentonite wherever either is needed. His suggestion of dual-purpose wells is a definite possibility in some locations, such as wells for industrial and irrigation purposes.

Electrical analogy models will certainly be helpful in analyzing dam problems as described by Horace A. Johnson. Fig. 13 clearly indicates the advantage of full penetration over partial penetration.

Professor Rutledge has made an excellent presentation of the variables and the effect of these variables on the design. He further emphasizes the importance of being certain that the well has adequate diameter to reduce friction losses to a minimum. H. H. Roberts and Carter V. Johnson have offered further data on the Fort Peck installation. Their suggestion to install a minimum number of wells initially is sound, and further emphasizes the flexibility of the well system provided the wells installed are large enough and deep enough. The failures mentioned involved the deterioration of the slotted black metal pipe used as an emergency installation during World War II. Recently, permanent installations were completed at Sardis and Fort Peck dams, using a slotted wood pipe.

Mr. Fahlquist has requested data on fabrication and installation of wells. Although experience with water well installations has been most helpful in selecting the general type of design, there is still much to be done in the selection of a well screen that can be depended upon for long life. Experience with relief wells has shown it to be a most serious problem. Usually the water carried by the pervious strata where pressure relief is required is much more highly charged with active salt than water-supply wells in the same area. At Sardis Dam metal well screens, which were giving excellent service in water-supply wells, failed within two years when operated as relief wells. The seepage water was highly acid, whereas in the water-supply wells it was neutral. At Fort Peck the ground water in the pervious strata was highly alkaline and was found unsatisfactory for water-supply purposes before the dam was built. The slotted black pipe used in the temporary installation during World War II started failing because of complete deterioration of the metal in about two years. At both of these dams slotted wood pipes have been used in the recent "permanent" installations. The writers recommend that only noncorrosive materials, such as wood, chemical resistant vitrified clay, stainless steel, monel metal, and other highly resistant metals should be used. Jetting to create continuous drainage is a definite possibility and should be further investigated. In this case the most difficult problem is to obtain a trench wide enough to discharge the required quantity of seepage.

Mr. Cedergren has properly emphasized the importance of exercising considerable care in constructing, and observing the functioning of, relief wells. In connection with the latter, the United States Corps of Engineers has a special research project on relief well design which includes detailed observations of all major relief well installations. A current meter has been designed for measuring the velocity in relief wells by lowering it into the casing. In this manner the total discharge of the well can be quickly obtained; and, in addition, by reading the velocity at different elevations, the quantity of seepage from the various strata can be determined.

Professor McNown has revealed one of the most troublesome factors in the presentation of mathematical solutions—that is, the lack of uniform symbols. The writers regret that the symbols used were not familiar to Professor McNown. Mr. Barron has presented an excellent discussion in which he has obtained a mathematical solution where a blanket of finite length is present. He has modified this solution for seepage through the blanket, which is an important factor in some cases.

Mr. Bennett's discussion should be especially helpful to engineers engaged in the design of relief well systems. He indicates the value of piezometer readings wherever they are available. Field pumping tests, of course, are highly desirable in the case mentioned, and in other cases where the area is highly developed and a large discharge is expected. However, pumping tests are expensive and must be conducted with experienced personnel and extreme care, or the results will be misleading. Where the quantity of seepage is not a major factor, it is usually cheaper to install additional wells than to conduct pumping tests. Modification of the formulas as presented by Mr. Bennett is certainly desirable where there is a definite gradient of flow away from the wells; but this is not often an important factor. As stated, jetting to form an outlet, as proposed by Mr. Fahlquist, may form a block for the more pervious strata and make seepage conditions in levee sections worse.

Mr. Mansur has stressed the importance of the type of well screen used in an installation. He places brass and cast iron in the preferred class. In some cases the use of these two materials may be highly questionable. His minimum diameter of 2.5 in. is considered entirely too small. It is the writers' opinion that this minimum should be 6 in. Quite correctly, Mr. Mansur emphasizes the danger of using dissimilar metals because of the rapid corrosion caused by electrolysis. Surging of relief wells is an essential part of well installation even though the well is gravel packed. Mr. Mansur has shown that wells behind levees will not increase the quantity of seepage by an exorbitant amount. The 27% increase given is reasonable where the pressure head at the land-side toe is reduced to 1.0 ft. Generally this great reduction of pressure is not necessary. Experience with large boils along the Mississippi River levees shows conclusively that a moderate increase in the normal seepage flow will prevent the occurrence of dangerous boils. During high water stages, the writers have seen numerous cases in which one or two large boils were giving adequate pressure release to an extensive area. However, when these boils were sacked (a ring levee built around them) to prevent movement of fines, other dangerous boils

occurred in the vicinity. In these cases one or two large relief wells (which, in effect, are controlled boils) would give permanent relief to such dangerous areas.

Reduction of well discharge rates after installation is a most important factor as shown by S. J. Johnson. The wells referred to by Mr. Johnson were too small and high velocities were a contributing factor in the reduction of flow. This reduction has not been evident on large adequately designed wells. However, clogging is another factor in favor of large-diameter wells and provisions for surging as a maintenance operation. The clogging of water-supply wells is more common because of the high drawdown heads which result in high entrance velocities.

Mr. Lane has added further to the mass of recorded experience in giving a specific example in which the seepage at the toe of a levee has proved dangerous. His discussion of the problem of determining where wells are needed should be most helpful.

Since mathematical design formulas for relief well systems are based on idealized asumptions as to the uniformity of the pervious stratum and blanket and the determination of permeability (which, at best, is only a good approximation), the results from the formulas should not be considered as an exact design but rather as an approximation to aid the judgment of the designer. With this thought in mind, it is the writers' opinion, that reasonably accurate results, quickly computed, are more desirable than mathematically exact solutions, achieved through repeated trials. Also, the writers wish to emphasize that, in general, wells of large effective radius are necessary to provide sufficient entrance face area and to keep well-screen and riser losses at a minimum.

The writers wish to express their appreciation for the most valuable discussions and to express their regrets that they did not have time to compare the various modifications of design with the procedure proposed in the paper, and with the actual installations.

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DISCUSSIONS

EFFECT OF STRESS DISTRIBUTION ON YIELD POINTS

Discussion

BY F. G. ERIC PETERSON

F. G. ERIC PETERSON,[12] Assoc. M. ASCE.—As stated by Mr. Shearwood, it is true that the value of the thesis presented "* * * will be realized through the promotion of investigation into the practical and useful application of the findings * * *."

The general formula (Eq. 12*b*) developed by Professor Gaylord, which gives a relation between the tension yield point and the bending yield point for various shapes of cross section, is quite ingenious and appears to give quite accurate results. It should be observed that, in shapes where a large proportion of the area of a cross section is far from the neutral axis (for example, wide-flange beams), the effect is quite small, whereas in cross sections with a large proportion of the area close to the neutral axis (rectangular, cylindrical, diamond shape, etc.) the bending yield point is appreciably greater than the tension yield point. The accidental eccentricity of 0.001 in. mentioned by Professor Gaylord may have existed. It would be difficult to avoid such a small eccentricity.

In the presentation of this paper the writer has not concerned himself with the behavior of the metal after the plastic state has been reached. In other words, he has visualized the final designed structure as acting elastically throughout (after using a suitable factor of safety) without being tied to a definite maximum stress. The relative dimensions of the structure have a definite bearing on the final maximum stress that will result. One structure may have a maximum stress of 30,000 lb per sq in., whereas another may have a maximum stress of 24,000 lb per sq in.; both may have the same factor of safety (N) as defined by the ratio of the ultimate failure load to the working load:

$$N = \frac{L_u}{L_w} \qquad (25)$$

NOTE.—This paper by F. G. Eric Peterson was published in April, 1946, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1946, by F. P. Shearwood, Edwin H. Gaylord, and L. J. Mensch; November, 1946, by George Winter, and O. Sidebottom and T. J. Dolan; December, 1946, by J. F. Baker and J. W. Roderick; and March, 1947, by R. K. Bernhard.

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Various structures investigated by the writer, using a factor of safety of $N = 2$, had final working stresses varying from about 22,000 lb per sq in. to about 30,000 lb per sq in.; but such stresses were always appreciably below even the tension yield point.

As a consequence of the study of limit design, the American Institute of Steel Construction has adopted a specification allowing a 20% increase in stress for continuous beams. Although this is probably safe, it is not logical, since, as mentioned herein, a given factor of safety based on ultimate failure loads does not result in the same final maximum stress for different structures.

Mr. Mensch's statement that for fixed beams with uniform load "* * * it would be stated that failure occurred under a bending moment of $w L^2/16$ * * *" is logical. This merely means that once the end cross sections have reached their ultimate resisting moments the beam acts as a simple beam until the center cross section has reached the same ultimate resisting moment. The end moment plus the center moment must equal $1/8\ w l^2$.

As stated by Professor Winter, "The problem is difficult to approach experimentally, chiefly because * * * strain measurements can no longer be translated into stress." However, consider the load-deflection diagram of a beam in bending: The first part of that diagram for steel is straight (OB', Fig. 10) and no other stress distribution than that shown in Fig. 2(*b*) can possibly be justified—that is, a straight-line stress distribution must exist.

Professor Winter states that in a "* * * cross section of a rectangular beam stressed partly into the plastic range * * * a shape more like that of Fig. 12(*c*) must be expected." The writer agrees with this statement but wishes to emphasize that this shape must be accompanied by the appearance of deformation wedges or Lueders lines. Professor Winter also states:

> "It is doubtful whether macroscopic Lueders lines can be detected when yield strains are of such small amount, although it is likely that slip bands in individual crystals could be observed in such zones by microscopic inspection."

In this connection it should be noted that the appearance of Lueders lines is not a gradual process but that they appear quite suddenly—to an appreciable depth at once, afterward expanding in a series of jumps as reported by C. F. Kollbrunner (44).[12a] Thus, the writer concludes that the process of initial yield is immediately apparent through the appearance of macroscopic Lueders lines.

Although the writer is not very familiar with the X-ray method of stress measurement, it is commonly agreed that it is not very accurate (possibly 10% in error). What seems more important in the present discussion is that this method sheds no light on the phenomenon of the upper yield point. It may be that the results obtained by the writer and others are directly associated with the existence of an upper yield point—that is, the upper yield point is the controlling factor when the stress is nonuniform and no fibers show any evidence of yield until the extreme fiber has reached the upper yield point.

Referring to Fig. 21 by Messrs. Baker and Roderick, it is quite evident that no yield would occur in any fiber until the extreme fiber reached the upper yield

[12a] Numerals in parentheses, thus: (44), refer to corresponding items in the Bibliography (see Appendix of the paper), and at the end of discussion in this issue.

point. Thus, the writer is quite ready to agree with Messrs. Baker and Roderick that the lower yield point is not necessarily raised but that the yield is governed by the upper yield point. Fig. 22 proves this contention. It is possible that the value of the upper yield point is affected by the shape of the cross section being tested.

An important question presents itself to the writer. In the usual commercial tension test the yield point that is reported is the lower yield point. Should this value be used in design, particularly where nonuniform stress is to result? If this lower yield point is the one that is always supplied by the commercial testing laboratories, it does not seem much amiss to refer to a "raised yield point" in bending or other nonuniform stress; or, perhaps, it should be given another designation. The usual testing machine allows the load to decrease because of the yielding of the specimen (especially in machines of a high elastic constant, as suggested by Professor Bernhard), thus exhibiting a lower yield point. Under field conditions of loading, the load is not decreased as the structure yields. It appears to the writer that it would be better if commercial testing machines were of a type that would not allow the load to decrease since (in general) this would conform more closely to the conditions which a designed structure undergoes in actual use.

The testing machine used by the writer in his tests was a 60,000-lb mechanical pendulum machine with a direct-reading dial indicator. The speed of testing for bending was about 0.02 in. per min, whereas for the tension tests it was well under 0.01 in. per min.

The writer deeply appreciates the time and effort expended by various writers in contributing discussions on this paper.

Bibliography.—

(44) **Schichtenweises Fliessen in Balken aus Baustahl,"** by C. F. Kollbrunner, *Memoirs*, International Assn. for Bridge and Structural Eng., Vol 3, 1935, p. 222.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

NEW FORMULAS FOR STRESSES IN CONCRETE PAVEMENTS OF AIRFIELDS

Discussion

BY EVAN P. BONE

EVAN P. BONE,[21] ESQ.—The comprehensive formulas given in this paper prove to be very useful in the new design problems of pavements for heavy aircraft. It is possible to predict stresses under conditions not heretofore determinable from purely theoretical analysis. The design curves for rigid pavements given in an engineering manual[22] are based on Eq. 13, inasmuch as it has been determined that edge stress resulting from edge loading is critical for airfield pavements. Eq. 17, which is more general, has been found particularly applicable in the latest design problems of multiple-wheel landing gears.

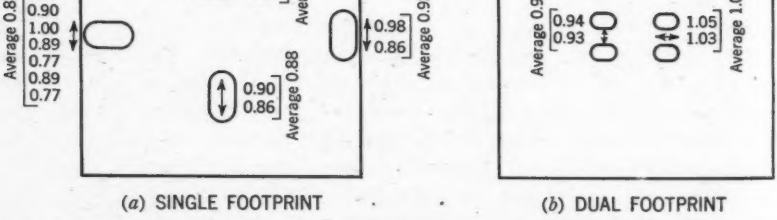

FIG. 2.—PLAN OF SLAB SHOWING RATIO OF MODEL EXPERIMENTAL STRESS TO WESTERGAARD THEORETICAL STRESS, WITH ARROWS INDICATING DIRECTIONS OF STRESS

By the use of this formula it is now possible to compute the edge stress resulting from the superimposition of stresses from two wheels of various spacing, when both are symmetrical with respect to the y-axis.

NOTE.—This paper by H. M. Westergaard was published in May, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1947, by Robert Horonjeff.

[21] Engr., Rigid Pavement Laboratory, Corps of Engrs., Dept. of the Army, Mariemont, Ohio.

[22] "Ad Interim Engineering Manual for War Department Construction," Corps of Engrs., Office of the Chf. of Engrs., July, 1946, Pt. XII, Chapter 3, p. 16 (published by the War Dept., Washington, D. C.).

As a rough check of the formulas used, experimental strain measurements were made on a scale model at the Rigid Pavement Laboratory of the Corps of Engineers under the direction of R. R. Philippe, Assoc. M. ASCE. The model consisted of a plaster slab on a sponge rubber subgrade. The horizontal dimensions, including the length and width of footprint, spacing of wheels (if multiple wheels), and the "radius of relative stiffness" were all proportioned to scale (approximately 1:20) corresponding to a chosen prototype. The scale value of the "radius of relative stiffness" was secured by use of Eq. 1, in which the terms E, μ, and k were taken as constants of the materials of which the model was constructed. The desired scale value of l was obtained by solving for h, which, for the plaster material used, equaled $\frac{1}{4}$ in. The subgrade modulus, k, of the sponge rubber was evaluated by measuring the volume displacement under a given load. The strains were measured by electrical gages of 1/8-in. active length.

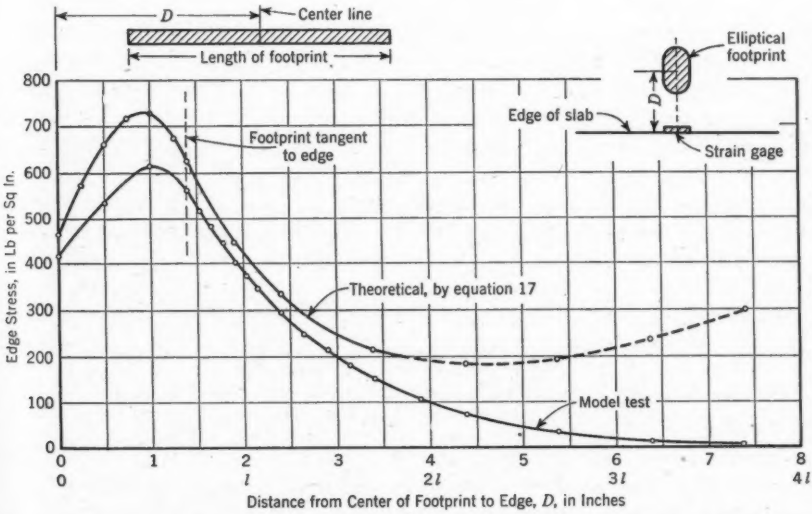

FIG. 3.—COMPARISON OF MODEL TEST SLAB STRESSES WITH WESTERGAARD THEORETICAL STRESSES (P = 40.42 LB; h = 0.265 IN.; a = 0.845 IN.; b = 1.410 IN.; $E = 2 \times 10^6$; μ = 0.2; k = 190; AND l = 2.0 IN.)

Fig. 2 gives the ratios of the experimental stresses to the theoretical ones. Several items from which the average values were determined refer to different conditions and locations of the respective types of loading, as, for instance, the edge stresses at all four edges of the same model slab. It is observed that the average experimental stresses are lower than the corresponding theoretical stresses. This may be due to experimental error alone, but there are indications that the difference was influenced by the fact that the sponge rubber subgrade acted more as a semi-infinite body than as a liquid support, which defines k.

Fig. 3 shows a comparison of experimental and theoretical edge stress as the load moves across a joint onto an adjacent panel, the joint having no capacity for load transfer. These experimental stresses are consistently lower than the theoretical ones; yet the similarity of the shapes of the curves and locations of the peak stresses are significant to a distance approximately $2l$ from the edge of the slab.

Although the model measurements were not performed with a high degree of precision, they may serve to carry convincing verification of the validity of the theoretical analysis.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

STABILITY OF SOIL SLOPES

Discussion

BY DONALD M. BURMISTER, AND EK-KHOO TAN

DONALD M. BURMISTER,[42] ASSOC. M. ASCE.—A distinctive contribution to the subject of stability of soil slopes has been made by the author—first, by presenting a new technique which may be used for investigating the broad field of stress displacement and failure phenomena in soils by the use of an "equivalent cohesion"; second, by providing an insight into the mechanics of failure of soil slopes and a better understanding of the conditions leading finally to failure; and, third, by interpreting such phenomena by photoelastic studies and by a mathematical treatment of the subject of failure in the plastic state. The writer wishes to consider in detail certain fundamental concepts involved in this study.

The first is the concept of cohesion. The author states in Part I (under the heading, "2. Phenomenon of Slides in Slopes of Cohesive Soils"):

> "Cohesion is essentially an intergranular stress phenomenon created in one case by capillary forces arising from the capillary moisture film at the grain contact * * *."

The truth of this statement should be obvious for sands and silts of cohesionless character. For example, the moisture films left in the fine sand at Daytona Beach, Fla., when the tide recedes converts the sand (most unstable when completely dry) to a firm hard surface, which makes an excellent race track. The importance of capillary cohesion can be judged from the fact that stable slopes in sands and silts of cohesionless character, but containing moisture films at the grain contacts, rarely have to be cut to angles flatter than about 45°. This cohesion is relatively permanent and can be relied on by engineers, otherwise all slopes in such materials would have to be cut to the angle of repose. The water in the contact moisture films does not evaporate entirely, or as readily as does free water, particularly if fine silt has been deposited at the grain contacts by rain water percolating downward through the deposit.

NOTE.—This paper by Ek-Khoo Tan was published in January, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1947, by Paul Baumann, and Gregory P. Tschebotarioff; June, 1947, by Jacob Feld, and Earl M. Buckingham; and September, 1947, by K. B. Hirashima.

[42] Associate Prof. of Civ. Eng., Columbia Univ., New York, N. Y.

The stress conditions induced by capillary forces acting in the contact films are shown in Fig. 19. Three zones are involved: A saturated zone below the ground-water level; a capillary zone extending to the upper limit of the capillary rise, in which interconnected capillary films form a continuous system through which capillary flow can take place; and a zone of condensation and evapora-

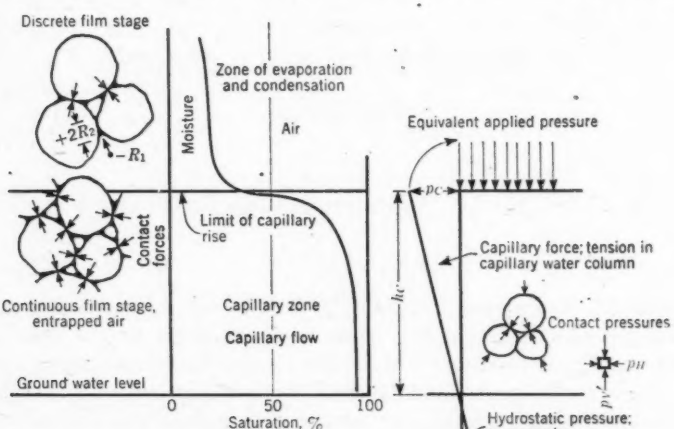

FIG. 19.—CAPILLARY FORCES AND INITIAL STRESSES IN COHESIONLESS SOILS

tion above the upper limit of the capillary rise, in which discrete water films occur at the grain contacts, but are not interconnected. The film relations are shown in Fig. 19(*a*); the state of stress in the contact water films is given by La Place's equation:

$$p_c = g\,\rho\,h_c = T_s\left(\frac{1}{R_1} = \frac{1}{R_2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

in which g is the acceleration of gravity; ρ is the mass density; h_c is the equivalent height of capillary rise; T_s is the surface tension of water; R_1 is the radius of the meniscus whose center is outside the film and is negative; and R_2 is the radius of the contact film. Eq. 29 does not limit the magnitude of either p_c or h_c, which in fine soils may considerably exceed the value for water in the normal state (30 ft of suction head).

Distribution of the moisture in the soil column, in percentages of that required for saturation, is shown in Fig. 19(*b*). The state of stress in the interconnected water films below the limit of capillary rise is shown in Fig. 19(*c*), the maximum tension in the water films occuring at the limit of capillary rise. This maximum stress condition in the water films must be exactly balanced by a stress condition in the grain structure of the soil—that is, by compressive stresses at the grain contacts. In the capillary region the stress conditions in the grain structure of the soil are identically the same as those that would be imposed either by applying a uniformly distributed vertical pressure at the

limit of capillary rise equal to the maximum tension, as shown, or by evacuating the region beneath a rubber membrane, which takes the place of the water films at the same level. The stress condition in the grain structure of the soil would be uniform with depth, but would not be isotropic, since the relation between the vertical and horizontal stresses is given by

$$p_h = p_v \left(\frac{\mu}{1 - \mu}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

in which p_h and p_v are, respectively, that part of the horizontal stress and the vertical stress due only to capillary forces at a point, and μ is Poisson's ratio, varying from 0.2 to 0.3 for sands. Above the limit of capillary rise, in the region of evaporation and condensation where the films are not interconnected, the stress conditions should be isotropic.

The most important fact is that under such circumstances an initial stress condition exists in the grain structure of the soil due to forces other than gravity. These stresses greatly increase the resistance of particles to relative displacements, particularly near the surface, where the gravitational forces may be small as compared with these initial stresses due to capillary forces. Therefore, the concept of an "equivalent cohesion" is applicable to such problems, because, as stated by the author, capillary cohesion is equal to the capillary force (or tension) times the tangent of the angle of friction.

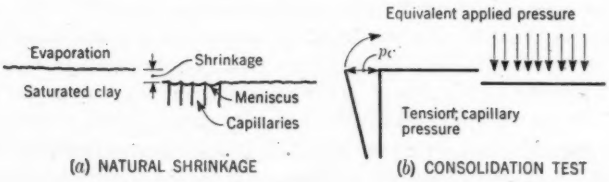

FIG. 20.—CAPILLARY FORCES AND INITIAL STRESS IN CLAY

It may not be so obvious that the concept is also applicable to clay-soils and clays. First, it is necessary to clearly separate two concepts: Cohesion, which is essentially an initial stress condition, whatever its source; and cohesive action, which involves plasticity and is quite another matter. The second concept will be deferred to the subsequent discussion on plasticity.

Conditions existing in a natural deposit of clay, exposed for the first time to evaporation forces by an excavation, are shown in Fig. 20. Evaporation of water from the surface of the clay results in an equal volume change in the region affected. Shrinkage of a mass of clay represents a strain, and hence the existence of a corresponding stress causing it must be recognized. This stress originates in the capillary meniscii formed at the surface of the shrinking clay by the evaporation processes, which bring into play active capillary forces at the surface of the clay, as shown in Fig. 20(*a*). The clay mass is subjected to a uniform stress. An identical situation is produced by applying an external pressure in a consolidation test, and the magnitude of the capillary pressure acting at any stage can be determined, inasmuch as a certain stress

must be associated with a certain strain for the given material. Shrinkage is really a compression of the clay under capillary forces. At the shrinkage limit, the resistance to further compression exceeds the maximum value of the capillary pressure that can be developed. Further evaporation merely causes the soil to dry out and causes air to enter the voids. Sufficient water (which cannot be completely evaporated at ordinary temperatures) remains in the soil as very fine capillary films to maintain this final stress condition, which may exceed 200 kg per sq cm.

If a series of clay specimens is prepared for unconfined compression tests—the first being tested immediately and the remainder tested at increasing intervals of time so that the water lost by slow evaporation produces approximately equal changes in water content and therefore in density—the strength of the specimens will show a regular increase with the increase in density. It may be properly reasoned, therefore, that cohesion is not an inherent, single-valued property of clays, as frequently thought, but is a function of density or water content. The really significant fact, however, is that the strength, and hence the cohesion, is a function of the initial stress conditions imposed on the specimen, whatever the source. Identical results would be obtained by preconsolidating each specimen under a series of increasing external pressures and permitting equilibrium to be reached by free drainage of the excess pore water. As soon as the specimen is removed from the consolidating device to be tested in unconfined compression, capillary forces will replace the external pressures, because only a very slight expansion is necessary.

In a natural deposit, the initial stress imposed by the action of capillary forces will be defined by Eq. 30 and will range between an isotropic state of stress when Poisson's ratio equals 0.5 and a ratio of the horizontal stresses to the vertical stresses of 0.667 when Possion's ratio equals 0.40. In some deposits of clays the material has been consolidated under much greater pressures than those imposed by the present overburden. In nature the initial state of stress is maintained under such conditions because of the impermeable nature of the clay, which does not readily permit absorption of water and hence volume increase, and because of the relatively small vertical expansion associated with the reduction of the vertical stress to that imposed by the present overburden. Therefore, the most significant fact about natural clay deposits is their initial stress condition, regardless of the source. It should be clear that, from the standpoint of the initial stress conditions, the new technique of imposing an "equivalent cohesion," or rather an initial stress condition, is a most valuable contribution for investigating and studying the field of stress, displacement, and failure phenomena in soils. The influence of the size of the mass of soil involved in the displacements or in failure cannot be learned from a small-scale model, but the model does give an insight into the mechanics of the phenomena and a better understanding of the stress displacement relationships and of the conditions leading to failure.

The second concept is that of plasticity, as a distinctive clay quality, which may be defined as the capacity to be deformed continuously under the action of a constant shearing stress without appreciable volume change. This quality is due primarily to the colloidal scale-like character of the true clay

minerals, and to the very impervious nature of clay. On the other hand, fine sand and silt are not plastic, because if a ball of the moist soil is shaken in the hand, the surface becomes wet and shiny, indicating a densification and a much more pervious character. If the ball is squeezed between the thumb and forefinger, the surface dries and water is sucked into the ball by the expansion of the soil under such action. In such a material the initial stress condition can change during deformations, because the stress in the pore water changes, becoming a positive hydorstatic stress with a compression and a negative stress or tension with a tendency for expansion. In clays, with their characteristic qualities and impervious nature, the initial stress condition is maintained unless the rate of deformation is exceedingly slow so that volume changes may occur. The dry sand model test with an imposed initial stress condition conforms more closely to the situation for a saturated clay than for a saturated fine sand or silt, particularly if Coulomb's equation for the shearing strength of clay—

$$S = c + N \tan \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

—can be considered to apply reasonably well. The chief difference is the magnitude of the angle of friction, ϕ, of clay, if such a factor exists for clays. The ratio of the horizontal to vertical stresses will be somewhat different, being of the order of from 0.667 to 0.875 for clay and of from 0.25 to 0.50 for sand.

As noted previously, plasticity is a distinctive characteristic of clay. Viscous or fluid flow is characteristic of liquids and suspensions and depends on only one property of the liquid, as expressed by

$$v \approx \frac{F}{\eta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

in which v is the velocity of flow; F is the force or head causing the flow; and η is the viscosity of the liquid or suspension. Plastic flow is basically different in one important respect. Bingham's[21] law states that plastic flow depends on two properties of the plastic material, such that

$$v \approx \frac{F - F_o}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

in which F_o is the yield value of the material, and λ is its consistency.

In viscous flow no static equilibrium is possible even under the slightest pressure or head. The significant fact about plastic flow is that, for stresses below the yield value, static equilibrium is possible and plastic flow or yielding ceases in time. If it were not for this important fact, it would be impossible to found any structure on clay. Below the yield value the flow is in the nature of a very slow plastic creep, which in most cases ceases after a time; above the yield value plastic flow can take place indefinitely at a constant rate under a constant shearing force and without appreciable volume change. The magnitude of the yield value is determined by the initial stress condition imposed on the clay at some time in its geologic history.

[21] "Fluidity and Plasticity," by E. C. Bingham, McGraw-Hill Book Co., Inc., New York, N. Y., 1922, Chapter VIII.

The third concept is that of the plastic state. It is necessary to differentiate clearly between plasticity as a clay-like quality (as previously defined), and the plastic state defined as a definite state of stress in the theory of plasticity without regard to the nature or the kind of material involved. A region can be said to have reached the plastic state when everywhere within the region the shearing stresses have become equal to the shearing strength of the material. This condition of plasticity has been expressed mathematically in Eq. 4, where the maximum shearing stress has become equal to the maximum shearing strength or cohesion, c. The maximum shearing stress can become equal to but cannot exceed the shearing strength of the material. This means that the plastic zone must spread to areas having a reserve of shearing strength until the total shearing load is carried—if such a condition is possible. In clays there will be a zone in which Eq. 4, the law of plastic flow, applies, and the failure will proceed essentially without volume change. In sands it is conceivable that flow failure may occur if a volume decrease occurs during the failure and the failure can take place rapidly.

Certain significant facts are shown in Figs. 3 and 4, and three regions may be defined. Above the final slip line is a plastic region in which the shearing stresses have become equal to the shearing strength. Below this is a transition zone, in which relatively large displacements have occurred, but below values required to mobilize the full shearing strength. Below this transition zone is an elastic zone, in which deformations are insignificant. Fig. 4 shows that, as the initial stress is increased, the angle of slope at failure being related to this initial stress through Fig. 2, the phenomenon changes from an essentially superficial one typical of cohesionless sand under very low values of the initial stress to one which is typical of the deep-seated character associated with slides in clays. The final sliding curve in each case appears to be localized at a definite depth, depending on the magnitude of the initial stress. The phenomenon shown in Fig. 3 is typical of the "toe failure slides," wherein the horizontal bottom of the excavation is at the level of the toe, or where a slide occurs in a weaker material in a bank. Usually this weaker soil is underlain by a higher strength material which limits the failure curve to the interface between the two materials. The phenomenon associated with a lubricated base (Fig. 9) is of interest because a number of rolled earth dams, built on foundations containing a plastic clay layer relatively near the surface, have shown considerable lateral movement during construction.

It is also significant to note that in the plasticity theory developed by the author the existence of an initial stress condition is an essential condition for deep-seated sliding failure curves, as shown in Fig. 12, and that the surface of the upper part of the slope is in tension (above point 3 in Fig. 12(a) and above point 5 in Fig. 12(b)). A fact of great practical importance is that failure is imminent when the strain in the embankment exceeds a certain rather definite limiting value for the given material. This is a consequence not only of the stress deformation characteristics of the soil, but also of the progressive failure phenomena due to the actual distribution of shearing stresses within an embankment and to the spreading of the plastic zone. These factors, together with the concentration of shearing stresses in the middle area of the rigid base

(see Fig. 9), may account for the wave pattern in the shearing strain (Fig. 5) and the corresponding stress pattern (Fig. 9), which are associated phenomena. In clays the phenomenon of progressive failure is likely to be more important than in sands (except in the dense state), because the stress-strain curves of clays usually show a characteristic large decrease in shearing strength after the maximum strength has been reached, when the test is made by strain control methods. This decrease in shearing strength is the result of disturbance effects resembling strain softening.

The fourth concept is that strains in soils and the stress conditions for failure depend exclusively on effective stresses, as stated by K. Terzaghi,[41] M. ASCE. The total stresses in a saturated soil consist of two parts. One part is the effective stress, which acts in the grain structure of the soil and causes compression; the other is the pressure in the pore water, which may be greater (or less) than the natural hydrostatic pressure, and even negative, depending on conditions. The shearing resistance that can be mobilized in a soil mass under any given circumstances depends only on the effective stresses. A change in stress in a saturated soil is always associated with a tendency toward change in its water content or volume of voids. If the soil tends to compress during shearing and the loading is rapid enough so that the water cannot escape within that short period, positive hydrostatic stresses are induced in the pore water in excess of the natural hydrostatic pressure. As a consequence, part of the load on a potential plane of shearing is carried by positive pore water pressures. The shearing resistance of the soil is then expressed by—

$$S = (N - u) \tan \phi \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

in which N is the total normal pressure acting on the potential surface or rupture or sliding; u is the pore water pressure (positive or negative excess over the natural hydrostatic pressure); $(N - u)$ is the effective stress (less than N, if u is positive and greater than N, if u is negative); and ϕ is the angle of friction.

The value of the pore water pressure, whether positive or negative, depends not only on the nature of the soil and its initial state with respect to density and to stress, but also on the rate of stress application, the permeability of the soil, and the size of the soil mass involved. The influence of the mass of soil involved in the displacement, or in the failure, cannot be learned from a small-scale model because the effect of mass in such phenomena primarily concerns the effective stresses acting throughout the mass. These stresses are determined as to distribution and magnitude by the drainage conditions and by the pore water pressures. The magnitude and distribution of the pore water pressures, whether positive or negative, depend not only on the initial density of the soil with respect to the critical density (the original concept stated by A. Casagrande,[42] M. ASCE), but also on the pressure on the potential sliding surface with respect to the pressure which is critical for the soil in the given density state (an extension of the original concept). Wherever the soil stresses are less than this critical pressure, the soil will tend to expand during shearing

[41] "Theoretical Soil Mechanics," by Karl Terzaghi, John Wiley & Sons, Inc., New York, N. Y., 1943.

[42] "Characteristics of Cohesionless Soils Affecting the Stability of Slopes and Earth Fills," by A. Casagrande, *Journal*, Boston Soc. of Civ. Engrs., Vol. 23, 1936, p. 13.

and the effective stress in Eq. 34, since u is negative, will be greater than the applied normal stresses. Thus, the soil will have greater shearing resistance if failure by a rapid slide is a possibility. Otherwise the failure will occur slowly, if at all, after the pore water pressures are relieved by drainage. On the other hand, whenever the stresses are greater than the critical value, failure is imminent, because the soil will tend to compress during shearing, and thus will induce positive pore water pressures. The effective stress in Eq. 34 will be considerably less than the applied normal pore water pressure stresses. It is probable that in a large structure certain regions may be in the critical state whereas adjacent regions are not, because in these adjacent regions the pressures are less than the critical. It is evident that a small-scale model can give no information on this phenomenon, because the stress conditions for the critical state cannot be satisfied, either relatively or as to distribution. These stability conditions are primarily a matter of laboratory determination of the shearing strength characteristics of the soils involved and of analytical analyses of the stability conditions. The only check is the behavior of full-scale structures. The small-scale model test on dry sand with controlled initial stress conditions does, however, give a valuable insight into the mechanics of the phenomenon and a better understanding of the stress displacement relationships and of the conditions leading to failure.

EK-KHOO TAN,[44] JUN. ASCE.—The writer is grateful for the discussions of his paper and for the interest shown in this subject. Mr. Feld's remarks on the concept of plasticity and failure of soil slopes are appreciated as they lead to a better understanding of the basic phenomenon. The writer has attempted to show by a small-scale sand model the possibility of sliding failure. The purpose of the tests is to acquire an insight into the actual mechanics of failure of large soil masses by slides. The present method of analysis is to assume a failure curve and then to calculate the factor of safety. If a plastic condition is initiated at any region in a soil slope that is overstressed locally and the region of overstress is allowed to spread, a progressive failure will eventually take place. Therefore, the factor to be considered in the design of slope, it seems, is to avoid regions of stress concentration.

Mr. Feld states that not all soils showing the superficial phenomenon are totally lacking in cohesion. The tests show that for cohesionless materials the failure could not be a deep-seated one, but is essentially a surface one. It does not preclude the case of surface movement of cohesive material where no deep-seated mass movement results in a slide.

In commenting on Fig. 2, Mr. Feld states that it would be more logical to extend the graphs for the loose and dense states to meet at 36°. At very low values of the vacuum, however, the restraining effect of the rubber membrane becomes rather important. It shifts the lines of failure for the loose and dense states parallel to themselves from an angle of repose of 36° maximum at actual zero equivalent cohesion to their positions shown in Fig. 2. The restraint effects are thus of the order of a vacuum of 0.55 in. of water, or 3 lb per sq ft, which represents an equivalent cohesion of 1.95 lb per sq ft.

[44] Shanghai, China.

Both Messrs. Baumann and Hirashima state that soil slope failure basically is due to pore water pressure. The writer agrees that this is an important factor contributing to failure of soil slopes. The pore water pressures developed in any given case will depend on the initial density and on the magnitude of the minor principal stress in relation to the critical values, which may vary from point to point in the mass of soil undergoing shearing deformations. The pore water pressure, conceivably, can be either positive or negative, depending on whether the soil tends to compress or to expand during shearing. Such relationships can only be studied in carefully controlled triaxial compression tests, in which the phenomena can be assumed to be sensibly uniform and homogeneous throughout the soil specimen. Introducing moisture conditions in the model studies would unduly complicate the situation. It was apparent that any attempt to simulate such phenomena in a model was impractical. Many more uncontrollable conditions also would be introduced, such as a lack of homogeneity, uniform density, and uniform moisture content; unknown failure conditions; and difficulty in obtaining actual failures in such models.

Mr. Baumann suggests the use of soil cement mixtures. Such a model would have to be a full-scale model, because the forces involved in the model used are relatively small, a vacuum of 1 in. of water representing an equivalent cohesion of only 3.4 lb per sq ft. Mr. Baumann also suggests that centrifuge methods would simulate cohesion better. It is obvious, however, that if the gravity forces are increased in a certain ratio the shearing resistance that can be mobilized is also increased in exactly the same ratio. Hence, if the slope were stable before centrifuging, it would remain stable during such a test at any increase in the mass or gravitational forces, excepting only if the model were tilted during the operation.

To overcome all the various difficulties, and to limit the number of variables, this new technique was developed by introducing an equivalent cohesion. Dry sand was used throughout all the tests. The sieve analysis of the sand used is:

Sieve No.	Percentage finer by weight
4	100
8	90
16	76
30	54
50	28
100	9
200	0

The maximum density was 116 lb per cu ft and the minimum, 99 lb per cu ft. The loose state used in the tests had a density of about 100 lb per cu ft and the dense state, one of about 110 lb per cu ft. The angle of friction in the loose state by shearing tests was 33°; in the dense state this angle was 41°.

As Mr. Feld points out, cohesion is a very complicated phenomenon taking into consideration the size of soil particles, intermolecular attraction, and capillarity of moisture films, as well as the interplay of dissimilar materials in intimate contact. The experiments on the soil slopes were set up to permit

a study of the influence of varying, equivalent cohesive strength in an ideal soil and to avoid all uncontrollable factors.

Messrs. Baumann and Buckingham suggest that the discrepancy in the analysis by the use of the Swedish circular arc method may be due to the fact that the angle of repose (33°) was used as the angle of internal friction. The angles of internal friction of some sands may vary between 27° for a loose state and 45° for a dense state. The 33° angle used in the computation was taken to be the average consistent with the density of the model sand, which was slightly above the loosest possible state. Tests on gelatin models indicate that the stress pattern in a soil slope is rather different from that assumed in the Swedish circular arc method of analysis. This conventional method is essentially one dealing with averages along the whole failure curve. The experiments have shown that failure is initiated through a local concentration of stresses. Spread of this concentration thus produces a slide.

Professor Tschebotarioff points out the importance of boundary conditions in the distribution of stresses within a soil mass. This is of particular importance when the soil mass is subjected to external applied pressure. The writer agrees with Professor Tschebotarioff that the rigid boundary as used in the test on gelatin models has an influence on the stress distribution of the model.

Mr. Buckingham stresses an important concept—that for clay deposits whose moisture content is prevented from escaping, thereby making consolidation of the deposit impossible, failure may take place when the deposit is subjected to a superimposed load. It is true that in such cases the laboratory shear test employed should approximate conditions in the field, and the quick unconsolidated shear test results should be employed.

Mr. Hirashima asks that additional data be supplied to make possible a comparison between the experimental and theoretical determination of the maximum height of the unsupported bank of cohesive soil that will stand vertically. The formula for the theoretical determination should read

$$H = \frac{4\,c}{w} \tan\left(45 + \frac{\phi}{2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

The restraining effect of the rubber membrane was found to be rather large. The curve for the loose state in Fig. 6 should be a straight line through the origin, approximately parallel to the experimental line. By producing the experimental line to the vertical axis, the effect of the restraint can be estimated to be approximately equivalent to 4 in. of height. Introducing the values of the unit weight and of the angle of friction into Eq. 35, and deducting the 4 in. gives values within the order of magnitude to be expected.

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DISCUSSIONS

STABILITY OF THIN CYLINDRICAL SHELLS IN TORSION

Discussion

BY R. L. MOORE

R. L. MOORE,[31] Assoc. M. ASCE.—Although consideration of the buckling characteristics of thin-walled cylinders in torsion has been limited largely to aircraft design, the solutions presented by Professor Sturm are of interest wherever thin curved plates are required to resist shear. The application of the theory within the range of proportions covered by Fig. 5 is relatively simple and a reasonably good agreement between observed and computed shear buckling stresses for tubes and thin cylinders has been shown. The purpose

TABLE 2.—COMPARISON OF OBSERVED AND COMPUTED SHEAR BUCKLING CHARACTERISTICS OF ALUMINUM ALLOY CYLINDERS HAVING RING STIFFENERS ONLY

Spacing of rings L_r, in inches	Ratio[a] $\frac{L_r}{D}$	Shear Buckling Stress, In Pounds per Square Inch		Wave Angle, α, In Degrees	
		Observed[b]	Computed[c]	Observed[b]	Computed[c]
35.62	1.18	1,110	1,500	14	11
27.00	0.90	1,280	1,750	18	12.5
13.50	0.45	1,710	2,560	25	17.5
9.00	0.30	2,570	3,170	30	21.5
8.81	0.29	2,310	3,220	30	22

[a] Mean diameter $D = 30.08$ in. and wall thickness $t = 0.020$ in. [b] The observed values are averages of three to five determinations for each ratio of L_r/D. [c] The computed values are based on the Sturm solution for simply supported ends.

of this discussion is to present a few additional data and thus to make some evaluation of the method proposed for predicting shear buckling in stiffened cylinders or curved panels.

NOTE.—This paper by R. G. Sturm was published in April, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1947, by S. B. Batdorf and Manuel Stein; November, 1947, by Glenn Murphy; and December, 1947, by L. H. Donnell.

[31] Research Structural Engr., Aluminum Research Laboratories, Aluminum Co. of America, New Kensington, Pa.

Tables 2 and 3 show comparisons between observed and computed shear buckling characteristics for aluminum alloy, circular cylinders having both circumferential and longitudinal stiffeners. These results, obtained in tests

TABLE 3.—COMPARISON OF OBSERVED AND COMPUTED SHEAR BUCKLING CHARACTERISTICS OF ALUMINUM ALLOY CIRCULAR CYLINDERS HAVING LONGITUDINAL STIFFENERS AS WELL AS RING STIFFENERS

Spacing of rings L_r, in inches	Ratio[a] $\frac{L_r}{D}$	Wall thickness t, in inches	Observed Values[b] for Different Combinations of Longitudinal Stiffeners						Computed[c] for No Longitudinals		
			No Longitudinals		Eight Longitudinals, 11.81 in. apart		Sixteen Longitudinals, 5.90 in. apart				
			Buckling stress	Wave angle, α	Buckling stress[b]	Wave angle, α	Buckling stress[b]	Wave angle, α	Wave angle, α	$L_r \tan \alpha$	Buckling stress
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
13.50	0.45	0.020	1,710	25	2,110	18	2,870	19	17.5	4.25	2,560
9.00	0.30	0.020	2,570	30	2,950	26	3,860	24	21.5	3.55	3,170
6.75	0.23	0.0205	....	....	....	....	4,740	28	25	3.14	3,860
4.31	0.14	0.020	....	....	4,130	35	6,750	....	32	2.69	4,660

[a] Mean diameter $D = 30.08$ in. [b] For specimens having longitudinal stiffeners, where tension field action prevented, the determination of single, well-defined values of buckling torque, observed stresses (in pounds per square inch) were based on the mean torques required to produce visible buckling in all panels of any one size. [c] Based on the Sturm solution for simply supported ends.

made during 1943 and 1944[32] at the Aluminum Research Laboratories (New Kensington, Pa.), are of interest because they involve ratios of diameter-to-wall thickness (D/t) greater than those considered by Professor Sturm in his comparisons, and because they show directly the influence of longitudinal stiffeners on shear buckling resistance.

Figs. 10 and 11 show the type of specimens used and indicate the manner of loading. The ends of the cylinders were fitted with steel bulkheads, which in turn were bolted to frames through which torque could be applied. One frame was fixed while the other (visible in Fig. 10) could be rotated about the central axis of the specimen. The cylinders were 30 in. in diameter and were formed of sheets 0.020 in. by 36 in. by 96 in., which were wrapped around the end bulkheads and riveted together in a simple lap joint. The ring stiffeners used on the inside were formed of ½-in. square bars, whereas the longitudinal stiffeners used on the outside were of U-section, formed of 0.032-in. sheet aluminum. All sheet and stiffeners were of 24S-T3 and T4 aluminum alloy having the following typical properties: Tensile yield strength, 46,000 lb per sq in. and tensile ultimate strength, 68,000 lb per sq in. It is evident from the magnitude of the stresses shown that buckling in the tests occurred well within the elastic range of the material.

A study of Table 2 reveals two interesting results: (1) The observed shear buckling stresses for cylinders having a D/t ratio of 1,500, with ring stiffeners

[32] "Torsion Tests of Stiffened Circular Cylinders," by R. L. Moore and C. Wescoat, *Advance Restricted Report No. 4E31*, National Advisory Committee for Aeronautics, Washington, D. C., May, 1944.

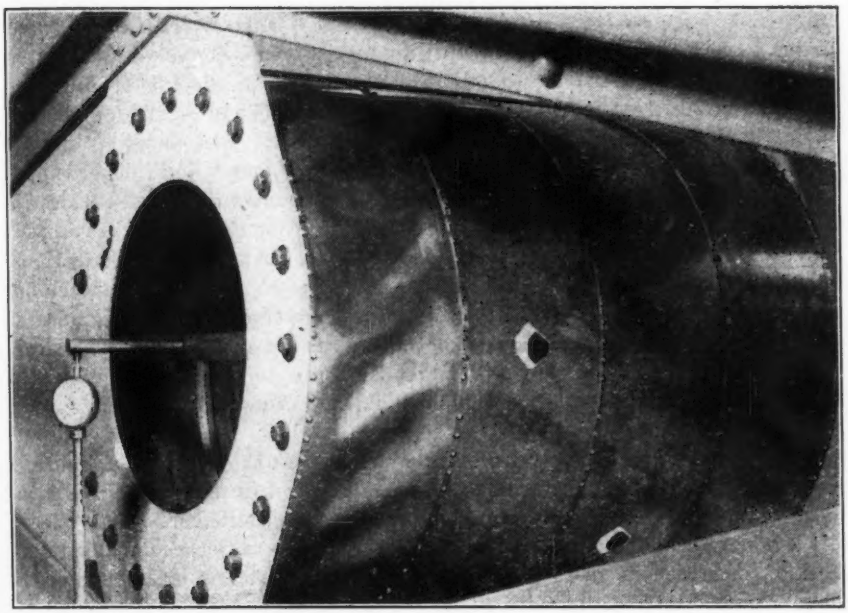

FIG. 10.—SHEAR-BUCKLING FAILURE OF CYLINDER HAVING RING STIFFENERS ONLY (y_r/D-RATIO OF BUCKLED PANEL = 0.29)

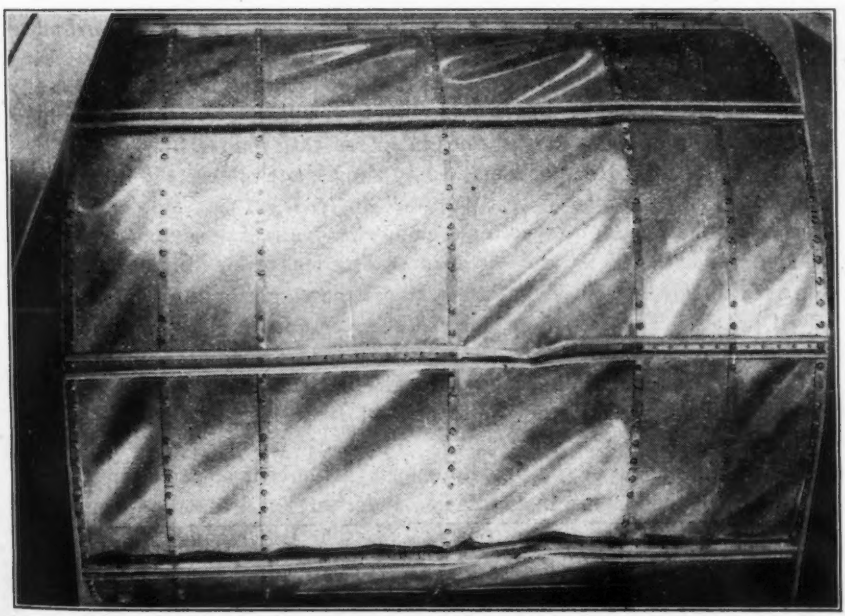

FIG. 11.—FAILURE OF STIFFENED CYLINDER UNDER TORSION LOADING CARRIED BEYOND THE POINT OF PANEL BUCKLING (y_r/D-RATIOS = 0.30 AT CENTER AND 0.14 AT ENDS)

only, ranged from 67% to 81% of the computed values; and (2) the wave angles α of the buckles, such as shown in Fig. 10, were in every case greater than those computed. The comparison from the standpoint of buckling stresses was only slightly less satisfactory than that shown in Fig. 7 for cylinders having a D/t-ratio of approximately 930. However, it was definitely not as favorable as that shown in Fig. 6 for relatively much longer and thicker tubes, having a D/t-ratio of only 138. Local out-of-roundness or initial buckles in the thinnest cylinders undoubtedly accounted for some of the discrepancies indicated between buckling stresses; but it is not so evident that imperfections of this kind were also responsible for the differences shown between measured and computed wave angles. In view of these observations it is significant that a considerably closer agreement between observed and computed buckling stresses might have been shown in Table 2 had the torsion theories of L. H. Donnell[5] or of S. B. Batdorf, Manuel Stein, and Murry Schildcrout,[25] Jun. ASCE, been used.

The experimental data given in Table 3 show that the addition of eight or sixteen equally spaced longitudinal stiffeners resulted in definite increases in shear buckling resistance over that observed for ring stiffeners alone. It is also evident in a few cases that a decrease in wave angle accompanied the addition of stiffeners, even though none of the angles was large enough to make a buckle intersection with such stiffeners unavoidable. This result has a direct bearing on the applicability of Professor Sturm's proposed method for determining the effect of longitudinal stiffeners on buckling resistance.

The computed wave angles shown in Table 3 for cylinders without longitudinal stiffeners were in every case less than those observed, even for cylinders with longitudinals. Moreover, since all these computed angles were less than required for an intersection of buckle waves and longitudinal stiffeners for any given spacing of rings, no increase in computed buckling resistance was indicated. This result was obviously not in accord with the tests.

The failure of the author's method to show more closely the influence of longitudinal stiffeners may be attributed to at least two factors—(1) the discrepancy between observed and computed wave angles for unstiffened cylinders; and (2) the failure to recognize the effect of longitudinal stiffeners, even at fairly wide spacings, on the buckling characteristics of the intermediate curved panels. Table 3 shows, for example, that for the tests in which the length-to-diameter ratio (L_r/D) was only 0.30 the wave angles were about 30° for the case of no longitudinals, 26° for eight longitudinals, and 24° for sixteen longitudinals. It would seem quite evident from Fig. 11 that these early shifts in angle were not the result of direct interference with longitudinal stiffeners. The angle of the buckles shown in the central panels in Fig. 11, after failure of the longitudinal stiffeners under tension field action, was about

[5] "Stability of Thin-Walled Tubes Under Torsion," by L. H. Donnell, *Technical Report No. 479*, National Advisory Committee for Aeronautics, Washington, D. C., 1933.

[25] "Critical Stress of Thin-Walled Cylinders in Torsion," by S. B. Batdorf, Manuel Stein, and Murry Schildcrout, *Technical Note No. 1344*, National Advisory Committee for Aeronautics, Washington, D. C., 1947.

33°, whereas the corresponding angles at first panel buckling were only about 26° and would show even less tendency to intersect the longitudinals.

From these few observations it appears that the effect of longitudinal stiffeners on the shear buckling resistance of curved panels is more far-reaching than can safely be determined on the basis of wave angles computed for unstiffened cylinders. Additional data are needed to investigate further the applicability of Professor Sturm's proposal, which, because of its simplicity, has considerable appeal from the standpoint of design.

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DISCUSSIONS

SEA LEVEL PLAN FOR PANAMA CANAL TO PROVIDE MAXIMUM SAFETY AND UNLIMITED CAPACITY

Discussion

BY FRED LAVIS, AND RAPHAEL G. KAZMANN

FRED LAVIS,[12] M. ASCE.—There seems to the writer to be no valid reason for considering a so-called sea level canal at Panama at this time. The difference in level between the Atlantic and Pacific oceans would always make one lock necessary. The present canal has not nearly reached its capacity in the thirty or so years it has been open. That capacity could be greatly increased by 24-hour operation. It is unlikely that traffic through the canal will increase more rapidly in the immediate future than it has in the past. The author gives few traffic data and no figures, that would seem to justify the proposed expenditure. A second set of locks would increase the capacity and would obviate the present restrictions of width.

The author confines himself to details of design and construction. No economic justification is offered for the large expenditure proposed. Increased costs of maintenance of the low-level channel and its accessories might offset savings in costs of operation of the present locks. It is doubtful whether a sea level canal would be more beneficial than the present canal. A sea level canal would be as vulnerable to modern high power bombs (not including atomic bombs) as is the present canal. The writer has seen the old channel of the Chagres River in Panama. He will readily acknowledge the ability of engineers, contractors, and modern excavating machinery, but he has a very high regard for the difficulties of the Culebra-Gaillard cut—more so, it would appear, than has the author of this paper.

RAPHAEL G. KAZMANN,[13] ASSOC. M. ASCE.—There has been a tendency in the past few years to justify a number of projects of dubious utility on the grounds of "national security." As applied to engineering projects this tendency does not favor objectivity and reliance on the merits of a project, when ably presented, to motivate its acceptance.

NOTE.—This paper by J. G. Clabyourn was published in February, 1947, *Proceedings*.

[12] Cons. Engr., New York, N. Y.

[13] Hydrologic Engr., Ranney Method Water Supplies, Inc., Columbus, Ohio.

In times of rapid and fundamental technologic change a "national defense" argument, which is the principal reason for a project, may become entirely invalid in a short space of time, thus nullifying the benefits to be gained by a proposed engineering improvement. The plan for a Panama Canal at sea level seems to be a "textbook" illustration of this—without detracting at all from the technical excellence and soundness of Mr. Claybourn's paper, if his basic assumptions are granted. The most important reason advanced by the advocates of a sea level Panama Canal is contained in the quotation,[4] cited by Mr. Claybourn in the last paragraph (under the heading, "Proposal for Conversion to Sea Level"):

> "If the hypotheses that a lock canal cannot be made impregnable and transits without delay of many months are essential to the security of the United States, are accepted, it must be assumed that a sea-level canal program must be the ultimate solution * * *."

It should be noted that this statement was made in December, 1944, six months before the first atomic bomb was detonated in the New Mexico desert. It is not clear to the writer just why anyone should use an atomic bomb on the canal when there are more profitable targets available such as industrial centers, reservoirs and lakes, and the internal transportation network. It is not clear, also, how the United States will identify the source of atomic bombs and then put the canal to work as a transportation medium. If such bombs were delivered by rocket no identifiable remains would be available for examamination after the explosion. In its sea level form, how would the canal add anything to the security of the United States during an A-bomb war that the nation does not already possess in the canal's present form?

Even assuming that the canal were a valid military target and thus subject to improvement on purely military grounds, would changing it from a lock canal to a sea level canal improve its security under an atomic attack?

There are two main possibilities—(1) that an A-bomb scores a direct hit on the canal somewhere between the present locks, but near one lock; and (2) that the bomb lands near one of the canal's entrances. Under the first possbility, in either form, the canal would be made inoperative for a long time. The lock canal would stand idle until new locks were built or until the old ones were rehabilitated. The sea level canal would be filled with poisonously radioactive sea water and any ships that were involved in the explosion (including all ships in transit at the time), even if they were not sunk, would be dangerously radioactive for many years. Proof? Some of the test ships at Bikini that remain afloat are still so radioactive that the United States Navy doesn't know how to dispose of them. Between radioactive sea water and poisonous fogs evaporating from the canal surface, it is difficult to imagine either type of canal being useful for several years after the explosion. Under the second possibility, assuming that the locks were not destroyed, the canal could remain open, since any radioactive sea water that fell in the canal could be removed by flushing with water from Gatun Lake. The sea level canal, on the other hand, would be as vulnerable to a near miss as it would be to a direct hit.

[4] "The Hypothetical Sea-Level Project," by E. E. Abbott, memorandum to the Engr. of Maintenance, the Panama Canal, Records Bureau, Balboa Heights, December 2, 1944.

The primary purpose of this discussion is not to show that military applications of atomic energy may make the entire "locks versus sea level" discussion meaningless. It is to point out that all engineers, whether or not they are planning projects involving "national security" would do well to add the Smyth report,[14] "One World or None,"[15] and "The Atomic Story"[16] to their shelf of reference volumes, after first studying them. Atomic energy and its possible applications should be considered in detail together with other technical and economic aspects of planned engineering work. Admittedly, this is extremely difficult, both because of the newness of this source of energy and specialized materiel, and of the secrecy connected with the development of atomic energy which seals the lips of those most familiar with its potentialities. On the other hand, it is difficult to understand how future expenditures in the field of civil engineering (for example, hydroelectric power and water supply) can be justified without a mature and detailed re-examination of accepted economic and social assumptions in the light of developments in the atomic field.

[14] "Atomic Energy for Military Purposes," by H. D. Smyth, Princeton Univ. Press, Princeton, N. J., 1945.

[15] "One World or None," a symposium edited by D. Masters and K. Way, McGraw-Hill Book Co., Inc., New York, N. Y., 1945.

[16] "The Atomic Story," by John W. Campbell, Henry Holt and Co., New York, N. Y., 1947.

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DISCUSSIONS

INVESTIGATION OF DRAINAGE RATES AFFECTING STABILITY OF EARTH DAMS

Discussion

BY HARRY R. CEDERGREN

HARRY R. CEDERGREN,[14] ASSOC. M. ASCE.—In this basic paper the author has introduced a subject of considerable interest to the designer of earth dams. Not only has he presented data regarding the development of pore pressures within several operating earth dams, but he has developed a mathematical analysis that should be of considerable value to research engineers. Very few data on the actual performance of earth dams have been made available to the engineering profession as a whole, and any contribution in this direction is a significant step forward.

Too often extremely elaborate investigations are made for the design of earth dams, but only meager observations (or none at all) are made of pore pressures within the completed structure. Frequently observation systems and schedules are set up and observed for a short time and then abandoned. It is difficult to convince many persons and agencies responsible for the operation of earth dams that relatively small sums of money spent in observing water pressures in earth dams and their foundations over long periods of time can pay dividends. In order that such observations may be of maximum value, careful field measurements of water pressures should be accompanied by dependable summaries of the soil properties having an important bearing on the rate of movement of the saturation line or changes in pressure within the water phase.

In engineering computations highly refined mathematical theory is often applied to situations in which the possible degree of accuracy of the mathematics is far greater than the limitations imposed in fitting the mathematics to real situations. Although the theory may be capable of a high degree of refinement, factors controlling the practical application of the problem generally can be determined only approximately. Nevertheless the theory can

NOTE.—This paper by F. H. Kellogg was published in September, 1947, *Proceedings*.

[14] With U. S. Corps of Engrs., Portland Dist., Portland, Ore.

be helpful in establishing probable limits within which the prototype can be expected to perform.

The author has based his mathematical derivation on Darcy's law and consideration of the quantity of water flowing out of an elemental volume of porous soil, with simplifications introduced to make his mathematical treatment possible. Since about 1940 the writer has used a graphical method based on Darcy's law and the flow net to obtain approximate solutions to certain nonsteady seepage problems encountered principally in earth dams. The method can be used in the study of falling saturation lines within embankments subjected to drawdown and other situations in which the position of a saturation line is changing. Although it must be considered only approximate, it is believed to have many practical applications.

Darcy's law provides a method of determining the velocity of a moving saturation line and, when used with the flow net, makes possible a study of the shape of a saturation line at successive positions in a structure. The flow net is derived for conditions of steady seepage, but in the following analysis it is assumed that the flow net can be used to obtain approximate solutions to certain nonsteady seepage problems by considering the moving seepage boundaries to be fixed momentarily. The analysis presented herein is based on the assumptions that no lag in flow or water pressure is caused by inertia or consolidation, and that changes in boundary conditions are transmitted quickly throughout the soil mass. To simplify the discussion, capillarity has been neglected. However, as indicated by the author, negative pore-water stresses, as set up by capillary forces in fine soils, cause an immediate lowering in the effective position of a falling saturation line by an amount not exceeding the effective capillary height. In this discussion, determination of the velocity of a saturation line will be reviewed prior to a discussion of application of the flow net in determining the shape of a saturation line at successive positions within a structure.

Darcy's law expresses the relationship between the average, or discharge velocity, v_D, the permeability of the soil, k, and the seepage gradient, i, as indicated by Eq. 37:

$$v_D = k\,i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37)$$

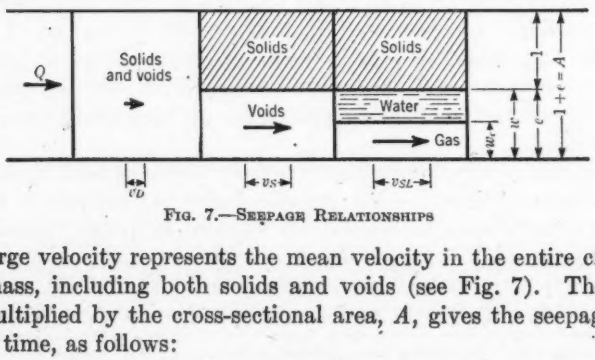

FIG. 7.—SEEPAGE RELATIONSHIPS

The discharge velocity represents the mean velocity in the entire cross section of a soil mass, including both solids and voids (see Fig. 7). The discharge velocity multiplied by the cross-sectional area, A, gives the seepage quantity per unit of time, as follows:

$$q = v_D\,A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

The effective velocity in the soil voids (the seepage velocity, v_S, Fig. 7) must be greater than the discharge velocity. Since the quantity of water flowing in the confined tube (Fig. 7) must be constant throughout the length of the tube, it follows that:

$$v_D\,(1 + e) = v_S\,e \quad \ldots\ldots (39a)$$

and

$$v_S = v_D\,\frac{1 + e}{e} = k\,i\,\frac{1 + e}{e} \quad \ldots\ldots (39b)$$

The seepage velocity, v_S, represents the effective velocity in the soil voids in the direction of flow lines for steady seepage conditions. When considering the rate of movement of a saturation line into or out of a soil mass, the quantity of water required to fill voids already partly filled, or the quantity of water draining out leaving the voids partly filled, must be considered. If the zone of saturation is moving into moist soil, less water must flow in to fill the voids than would be required if all the voids were empty. Likewise, if the zone of saturation is receding in a soil mass, leaving part of the voids filled, the saturation line can drop more rapidly than if all water must be removed. This factor was discussed by the author in terms of "drainability."

The velocity of a moving saturation line (v_{SL}, Fig. 7) is related to the seepage velocity in the following manner:

$$v_{SL}\,w_c = v_S\,w \quad \ldots\ldots (40a)$$

and

$$v_{SL} = v_S\,\frac{w}{w_c} = k\,i\,\frac{1 + e}{e}\,\frac{w}{w_c} \quad \ldots\ldots (40b)$$

Grouping all terms dependent on the quantity of water flowing into or out of a soil mass in a drainage factor, C, Eq. 40*b* is simplified as follows:

$$v_{SL} = k\,i\,C \quad \ldots\ldots (41)$$

in which

$$C = \frac{1 + e}{e}\,\frac{w}{w_c} \quad \ldots\ldots (42)$$

In Eq. 42 w is the saturated water content and w_c is the change in water content caused by a change from the moist to the saturated state, or vice versa.

From Eq. 41 the velocity of a moving saturation line can be estimated at any point at which the seepage gradient is known, provided the permeability and drainage factor are known. Knowing the velocity of the saturation line, the time required for a given length of movement may be determined from

$$L = v_{SL}\,t \quad \ldots\ldots (43)$$

in which L is the distance traveled in time, t, and v_{SL} is the velocity of the saturation line. For conditions existing during the drainage of water out of an embankment, or during saturation of an embankment, the seepage gradients are in a state of flux, and the seepage velocities are constantly changing. A rigorous solution for these conditions would be highly complex, but approximate solutions can be made by determining the velocity at finite increments of

distance in a structure, and summing up the times required for all increments, as follows:

$$\Delta L = v_{SL}\,\Delta t \quad \ldots\ldots (44a)$$

and

$$L = \Sigma v_{SL}\,\Delta t \quad \ldots\ldots (44b)$$

By use of Eqs. 41 and 44*a* studies may be made of the velocity of a moving saturation line, the time required for a given length of movement, or the distance moved in a given time. Studies of this kind for earth dams can be made in conjunction with determination of the shape of a moving saturation line with the flow net. From the flow net the relative velocities of different points along a moving saturation line at any given moment can be approximated, since the velocities are proportional to the seepage gradient, i, and the gradient can be determined at any point in a flow net from the following fundamental relationship:

$$i = \frac{\Delta H}{\Delta L} \quad \ldots\ldots (45)$$

in which ΔH is the drop in head in distance ΔL in the direction of flow.

Application of the foregoing principle to the drainage from an embankment subjected to drawdown is illustrated by an analysis of the cross section in Figs. 8 and 9, which show the same shape as the model in Fig. 2. It is assumed that the section has been subjected to steady seepage with a full reservoir

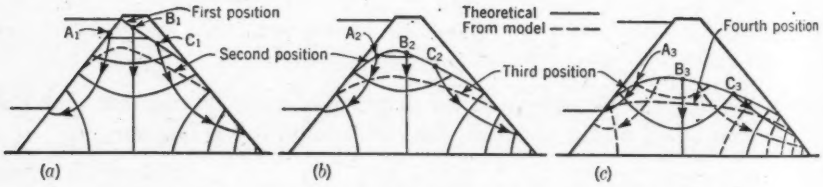

FIG. 8.—FLOW NETS FOR VARIOUS POSITIONS OF THE SATURATION LINE

until equilibrium has been established, and the reservoir was lowered instantly to the minimum pool level. Recognizing that the flow net may only represent the approximate true seepage conditions within embankments subjected to nonsteady seepage, a flow net is drawn for the initial steady seepage boundaries (Fig. 8(*a*)). The seepage gradient is then determined at a number of points along the saturation line (points A_1, B_1, and C_1, Fig. 8(*a*)), and the saturation line is moved ahead a moderate amount, with the relative distances moved proportional to the seepage gradients at the various points along the original line. A new flow net is constructed for the new saturation line (Fig. 8(*b*)), a new set of gradients determined, and the saturation line moved to a third position shown by a dashed line in Fig. 8(*b*). This procedure is repeated a sufficient number of times to define the shape of the saturation line over the range desired. Having established the position of the saturation line at a number of intervals, the time required for a given length of movement can be examined in any part of the drawdown zone with the aid of Eqs. 41 and 44*a*. When using Eq. 44*a*, Δt is the time required for movement of the saturation

line a distance ΔL, as for example—from the first to the second position in Fig. 8. This procedure, illustrated by a determination of the theoretical time required for the saturation line to move vertically downward at the center of

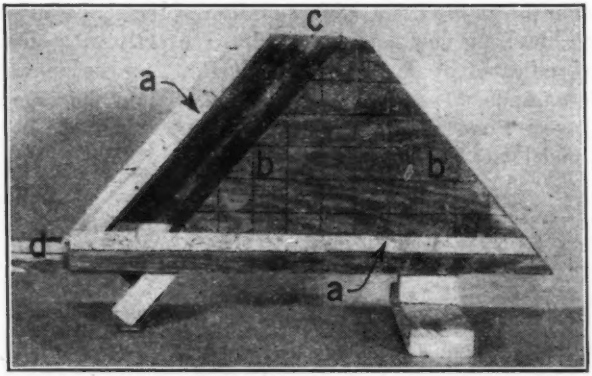

FIG. 9.—SIMPLE VISCOUS FLUID MODEL USED IN DRAWDOWN TESTS

the section in Fig. 8, is shown in Fig. 10. The seepage gradient at the center line is first determined for each position of the saturation line for which a flow net has been constructed, and the downward velocity of the saturation line at the center line computed from Eq. 41, using a drainage factor, C, and a permeability coefficient, k, which have been determined to be representative of the cross section. The velocity and time curves in Fig. 10 were computed

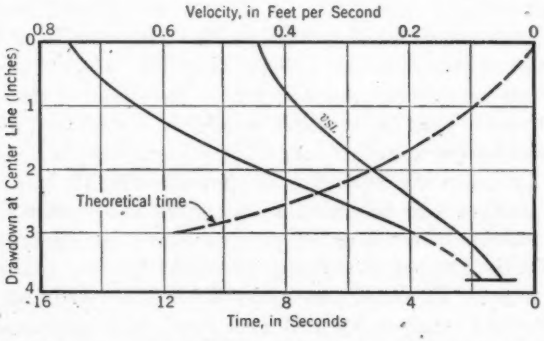

FIG. 10.—DETERMINATION OF THE THEORETICAL TIME FOR DRAINAGE FROM A MODEL

for a small, viscous fluid model having the same shape as the cross section in Fig. 8 (see Fig. 9). The base width of the model was $10\frac{1}{4}$ in.

Viscous fluid models provide a valuable supplement to the flow net in studying moving saturation lines. The model shown in Fig. 9 was constructed from plywood and the shape of the saturation line and its direction of movement compared with theoretical determination of these factors. The plywood cross section designated b–b, Fig. 9, is separated (by strips a) about 0.1 in. from a

glass plate that can be clamped on the front side of the model (not shown in photograph). The reservoir (c) is filled with a relatively viscous fluid, such as glycerine, which flows through the narrow space simulating flow through an embankment. Drawdown of the reservoir in the model is readily produced by pulling the slide (d) at the left. Determination of the shape and direction of the saturation line from flow nets agreed almost exactly with observations of the model, as indicated by Figs. 11(*a*) and 11(*b*). To the knowledge of the writer, A. Casagrande, M. ASCE, first proposed viscous fluid models for the study of seepage. Professor Casagrande and W. D. Shannon, Jr., have used this type of model extensively in nonsteady seepage studies.

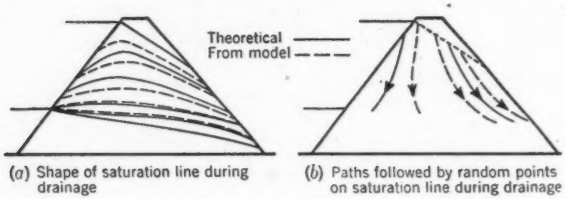
Theoretical
From model
(*a*) Shape of saturation line during drainage
(*b*) Paths followed by random points on saturation line during drainage

FIG. 11.—DETERMINATION OF THE SHAPE AND DIRECTION OF THE SATURATION LINE

The principles outlined herein are useful not only in studying changes in the position of a saturation line following reservoir drawdown, but also in obtaining approximate solutions to many other nonsteady seepage conditions. If water is flowing into a soil mass (as, for example, during saturation of an embankment), the same basic method[15] can be used; but the drainage factor, C, should be computed using w_c to designate the increase in water content accompanying a change from a moist to a saturated condition.

The author has suggested that high quality materials have been used for the protection of upstream slopes in some instances where less pervious, and cheaper materials may have been adequate. Recognizing this possibility, it is true, nevertheless, that under some conditions a blanket of high quality, pervious material on the upstream face of an embankment is a valuable investment. If one considers the embankment shown in Fig. 12, it is evident that a free draining blanket can substantially improve the stability of upstream slopes of embankments constructed predominately of impervious soil. In some instances, the improved stability provided by free draining, granular material may pay for its extra cost many times. For example, in locations requiring expensive tunnels to by-pass river flow, or for permanent operation, steepened embankment slopes (permissible with the use of free draining blanket material) may permit substantial reduction in the cost of the outlet works. In such locations, a stable, free draining blanket may be a highly valuable investment. To be of maximum benefit, the blanket should be free draining to the extent that the saturation line can closely follow a lowering reservoir. An approximate solution for this special drawdown problem was developed by the writer in 1940 using the aforementioned principles.

[15] *Civil Engineering*, August, 1941, p. 499.

In studying drainage out of thin blankets similar to that shown in Fig. 12, the criterion was established that the saturation line must closely follow the lowering reservoir, lagging on an average slope, S, that remains essentially constant during drawdown. The slope, S, may be established arbitrarily, or it may be the maximum allowable slope of the saturation line established by stability analysis. On the basis of the foregoing criterion a flow net and

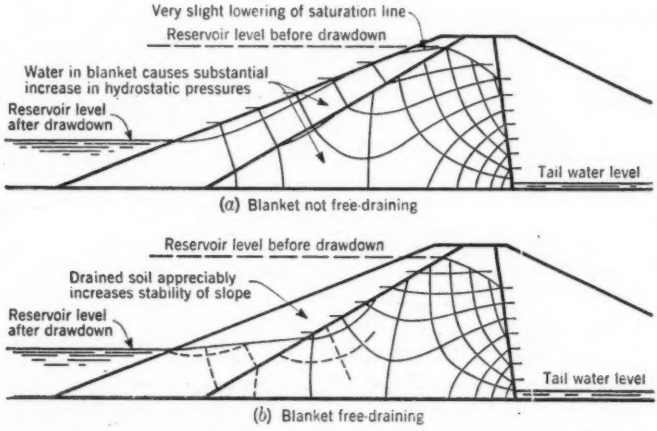
Very slight lowering of saturation line
Reservoir level before drawdown
Water in blanket causes substantial increase in hydrostatic pressures
Reservoir level after drawdown
Tail water level
(a) Blanket not free-draining
Reservoir level before drawdown
Drained soil appreciably increases stability of slope
Reservoir level after drawdown
Tail water level
(b) Blanket free-draining

FIG. 12.—INFLUENCE OF PERMEABILITY OF BLANKET ON THE STABILITY OF THE UPSTREAM SLOPE OF A DAM AFTER RAPID DRAWDOWN

viscous fluid model study was made. These studies led to the conclusion that simplified conditions (see Fig. 13) could be used in obtaining reasonably accurate solutions to this problem. In the first place, it was found that a straight line could be substituted with little error, for the slightly curved saturation line, and the slope of this line could be used as a measure of the average seepage gradient within the blanket with little error. Also, for thin blankets similar to those shown in Figs. 12 and 13, the mean direction of flow within the blanket is approximately coincident with a line bisecting the two sloping boundaries of the blanket (for example, line x–x, Fig. 13), intersecting the horizontal base line at an angle α. Substituting these simplified conditions, the velocity of the saturation line, v_{SL}, in the direction of line x–x may be determined from the following equation derived from Eq. 41:

$$v_{SL} = k\,S\,C \qquad \text{or} \qquad k = \frac{v_{SL}}{S\,C} \ldots\ldots\ldots\ldots\ldots\ldots (46)$$

in which k is the coefficient of permeability of the soil and S is the average slope of the saturation line. The relationships involved are illustrated by the diagrams in Fig. 13.

Generally the maximum velocity of drawdown is established by the discharge capacity of outlet works, and it is desired to determine the minimum permeability of a protective blanket that will permit the saturation line to drop within a blanket, closely following the elevation of the reservoir. The

velocity of drawdown of the reservoir, v_R, is related to the mean velocity of the saturation line, v_{SL}, in the following manner (see Fig. 13(*b*)):

$$v_{SL} = \frac{v_R}{\sin \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

Hence, the permeability that will permit the saturation line to follow the reservoir, with a specified small slope, S, can be approximated by combining Eqs. 46 and 47 as follows:

$$k = \frac{v_{SL}}{S\,C} = \frac{v_R}{S\,C \sin \alpha} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48)$$

Eq. 48 is proposed only as an approximate solution for the relationship between permeability and drawdown velocity for blankets having the general

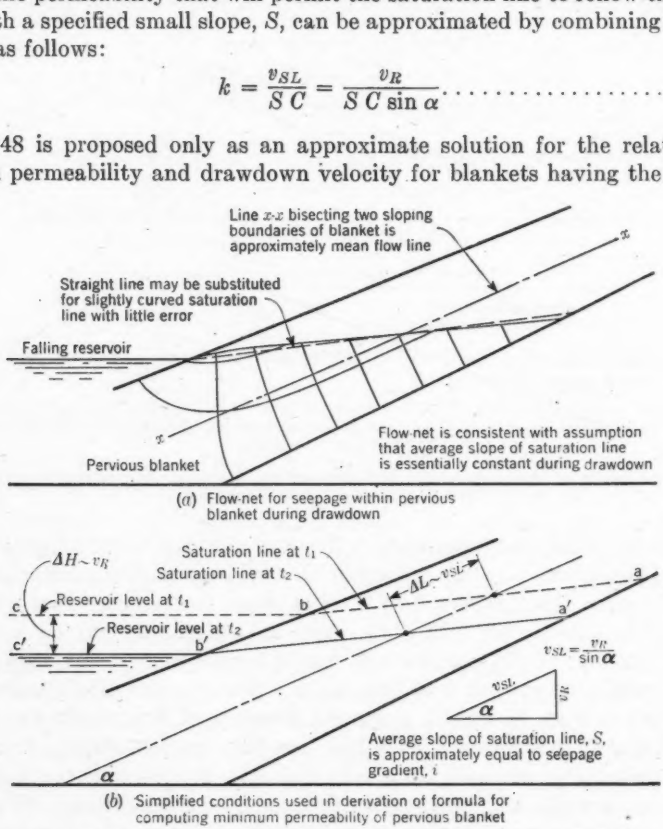
Line x-x bisecting two sloping boundaries of blanket is approximately mean flow line
x
Straight line may be substituted for slightly curved saturation line with little error
Falling reservoir
x
Pervious blanket
Flow-net is consistent with assumption that average slope of saturation line is essentially constant during drawdown
(a) Flow-net for seepage within pervious blanket during drawdown
ΔH ~ v_R
Saturation line at t1
Saturation line at t2
ΔL ~ v_SL
a
c
Reservoir level at t1
b
a'
c'
Reservoir level at t2
b'
v_SL = v_R / sin α
v_SL
v_R
α
Average slope of saturation line, S, is approximately equal to seepage gradient, i
α
(b) Simplified conditions used in derivation of formula for computing minimum permeability of pervious blanket

FIG. 13.—RELATIONSHIPS USED IN DETERMINING MINIMUM PERMEABILITY

shape shown in Fig. 12. This expression provides a quick method of determining the approximate permeability required if the maximum benefit is to be derived from protective blankets. If available materials are considerably less pervious than the minimum calculated from Eq. 48, drainage from the blanket can be neglected in stability studies. If the available materials, either natural or processed, are several times more pervious than the minimum indicated, the improved conditions within the blanket, as typified by Fig. 12(*b*), should be considered in the stability analysis. For cases in which the permeability of available materials is either several times more pervious than required, or several times less pervious, errors inherent in the method should be rela-

tively unimportant. Hence, an approximate solution such as Eq. 48 should have many useful applications. Doubtful cases will require individual analysis.

The graphical method outlined herein for the approximate analysis of moving saturation lines requires a knowledge of the construction of flow nets. However, this requirement should be no serious handicap, as reasonable skill in their construction can ordinarily be developed with moderate effort. New situations may require appreciable time for the development of satisfactory solutions; but once the general flow pattern for a given type of condition has been established additional flow nets are obtained with diminishing effort. After an engineer has developed reasonable skill in the construction of flow nets he finds that many apparently new problems are closely related to situations with which he is already familiar; hence, he encounters fewer and fewer new conditions. Also, if he studies accurate flow nets constructed by other engineers he can benefit considerably.

The author has stated that the probable danger of major slides of upstream slopes of dams subjected to rapid drawdown has been overemphasized, and he has indicated the manner in which capillary forces may cause a reduction in pore pressures in earth dams constructed of fine-grained soils. Although laboratory experiments may be helpful in establishing the probable effective height of the capillary zone in proposed embankment materials, observation of actual dams under operating conditions is believed the final answer to the study of this factor. Also, pore pressures caused by the sudden increase in effective weight of large soil masses in embankments subjected to drawdown are factors about which few factual data have been published. Dependable observational data for existing earth dams are needed for an evaluation of the combined result of the many factors effecting pore pressures in embankments.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

MISSISSIPPI RIVER CUTOFFS

Discussion

BY GERARD H. MATTHES

GERARD H. MATTHES,[20] HON. M. ASCE.—The discussions of this paper have served to widen the scope of the original subject matter and have been valuable in focusing attention on certain important phases of the cutoff problems not covered in the paper, as well as in calling for additional information.

Early advocates of cutoffs appear to have been numerous; among them were laymen who had become thoroughly acquainted with the Mississippi River and its modus operandi. The late Mr. Pharr, in his discussion, called attention to the plan proposed by James B. Miles in 1897. It showed a remarkably clear conception of what might be accomplished by making twenty-six cutoffs. An even earlier proposal, noteworthy for revealing the problems and trends of thought of that time, appeared in the *Weekly Vicksburg Whig* of December 28, 1859. It evidently was prompted by the occurrences of the great flood of 1858, which was followed by two more floods in 1859. It is quoted in full:

> "Cut-Offs—To the Members of the Mississippi Legislature—Gentlemen: —The subject of a general system of cutoffs on the Mississippi River, is rapidly gaining interest throughout the whole swamp country. In making laws upon this subject, the great object is to adopt a general system and not legislate for any special locality. The statute of Mississippi making it a penal offense to make cutoffs would be advocated upon the ground, that an isolated cutoff disconnected with the general system will do a great injury to the planters on either bank of the river—immediately below—nine-tenths of swamp planters are perfectly willing for such a law to remain on the statute books of both Mississippi and Louisiana—provided the legislatures of the two states make laws for a general system of cutoffs to commence below, and made whenever practicable along the river throughout the two states. I defy anyone to bring forward the names of twenty respectable swamp planters who would oppose such a measure. CATO
>
> Madison Parish, Louisiana, Dec. 19, 1859."

NOTE.—This paper by Gerard H. Matthes was published in January, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1947, by W. E. Elam, C. L. Hall, H D. Vogel, and Harry N. Pharr; June, 1947, by Lytle Brown, and Anson Marston; and September, 1947, by E. J. Williams, Jr., and Eugene A. Graves.

[20] New York, N. Y.; formerly Head Engr., Mississippi River Comm., and Director, U. S. Waterways Experiment Station, Vicksburg, Miss.

These two proposals show that the fruitlessness of making local changes in the river's course had been recognized early. The necessity for making continuous any corrective treatment by channel shortening was obvious. However sound, and however well accepted by many engineers, the magnitude of any comprehensive plan required deferment because of a lack of suitable dredging equipment and the great risks involved. Moreover, as pointed out by Mr. Pharr and by Major-General Brown, prior to 1917 federal appropriations were restricted to works in aid of navigation; flood control by use of cutoffs and levees had not then been recognized as a proper federal function. It seems appropriate to mention here that the first powerful dredge built specifically for use on the Mississippi River was designed by a member of the Mississippi River Commission, the late Henry Flad, Past-President, ASCE, who personally supervised its construction (1891–1892).[21]

Dimming recollections of the events that took place fifteen years ago apparently have caused conflicting statements to creep into some of the discussions. For the sake of historic accuracy, an effort is made herein to set divergent opinions at rest by reference to the writer's personal diaries, which fix dates and events beyond controversy. These show that the first comprehensive study of the cutoffs on the Lower Mississippi River dates back to October 24, 1930. The writer, then employed at the United States District Engineer's Office in Norfolk, Va., in charge of surveys[22] (navigation, water power, flood control, and irrigation) of certain rivers in Virginia and North Carolina, was called by Maj.-Gen. Harley B. Ferguson, M. ASCE, then Colonel and Division Engineer, to the South Atlantic Division Office at Norfolk to assist him with starting this study. It was aimed at effecting major flood-stage reductions in the Lower Mississippi River by cutoffs, and also contemplated stabilization of the new channel by the revetment of banks.

General Ferguson's conception was a bold one and without precedent for such a large meandering river. Fundamentally, however, his plan was sound, as it was aimed at preserving hydraulic gradients so as not to exceed those prevailing in stretches where the river was known to possess an "orderly channel." This term was applied by General Ferguson to designate a channel having mild curvature, free from islands and back channels, and whose shifting tendencies involved no excessive bank caving and bar building. Fundamentally sound, also, was that part of his plan which called for promoting the lowering of the river bed (and therefore the low-water profile) sufficiently, so that low-water depths needed by navigation would not become reduced. In these respects General Ferguson set up criteria which, so far as is known to the writer, had never before been promulgated in projects contemplating river shortening by cutoffs. Certainly, none of the earlier advocates of cutoffs for the Mississippi River had ever insisted on, or even suggested, the importance of lowering the river bed as an integral part of any cutoff program, or of keeping water surface slopes within the range of natural slopes.

He recognized that the energy developed by a body of water flowing at the rate of 1,000,000 cu ft per sec (which at that time represented the bankfull

[21] "Early Presidents of the Society," *Civil Engineering*, July, 1937, p. 529.
[22] *House Document No. 308*, 39th Cong., 1st Session, 1926.

discharge of the river) on gradients of only from $\frac{1}{4}$ ft to $\frac{1}{3}$ ft fall per mile was of tremendous magnitude; any work planned without regard to maintaining proper slopes for both bed and water surface could create conditions which, once out of hand, might well become uncontrollable. Therefore, his criteria amounted (physically) to the necessity of reconstructing the river channel at a lower elevation. To this end he planned his operations with a view to enlisting to the fullest possible extent the river's own energy in enlarging the cutoff channels and also in degrading its bed, assisting this with adequate dredging wherever clay beds or highly compacted silts resisted erosion. In 1930 a great deal of thought was devoted by him to the types of dredges that would be particularly suited to these phases of work, and by 1932 two special dredges were being outfitted. Profiles of the river bed had been prepared to study the characteristics of the succession of pools and crossing bars typical of the river.

Subsequent events have since proved the correctness of General Ferguson's judgment in regard to the ability of the Lower Mississippi River to erode its bed where it is composed of sand, and to alter its curvature in response to slight changes in hydraulic gradient. It is deemed pertinent to state that the river actually has shown itself sensitive to changes in gradient as slight as 0.01 ft per mile. This had been perceived when the early office computations in Norfolk were made. Slope figures from the start were carried out to three decimal places, and the practice developed there was generally adopted later in the office of the President of the Mississippi River Commission. The cutoff program from its inception required continuous study of slopes throughout the period that General Ferguson was connected with it. To implement this study, General Ferguson, after he became President of the Mississippi River Commission, ordered gages maintained at 5-mile intervals throughout the many hundred miles of river affected by his operations. The cost and labor involved in maintaining these gages (all referenced to sea level datum), the reading of them twice a week or more often, and the computing of the slopes, represented a large investment that was felt indispensable in order that nothing be left to chance.

The chronology shows that by November 22, 1930, General Ferguson had completed for submission to the Board of Rivers and Harbors a 38-page memorandum report recommending cutoffs in the 370-mile stretch between the mouths of White River and Old River. It included provisions for extensive revetting of banks which were deemed in need of protection against the effects of the increased energy released by slope steepening. This report received consideration by the Board of Rivers and Harbors at its sessions on December 2, 1930, and January 5 and February 3, 1931 (which the writer was required to attend); on the latter date the board approved General Ferguson's report, including his recommendation for undertaking a certain amount of experimental dredging aimed at expediting the lowering of the river bed.

At this time there had developed among engineers two distinct schools of thought—those who favored cutoffs, but who felt that extended flood-stage lowerings could result only when a reasonable amount of bed lowering could be brought about, and those who opposed cutoffs, because they felt the river would not lower its bed acceptably, and that, therefore, any flood-stage lowering effected by a cutoff would soon dwindle to zero upstreamward in accordance

with customary backwater computation. Col. E. H. Schulz, M. ASCE, a member of the board, went to considerable trouble to prove this limitation mathematically to the board at its session on January 5. Also, in this second school of thought were many of the engineers—both military and civil—stationed on the Mississippi River. Finally, many engineers insisted that the meandering habits of the river would soon restore the river to its former length and thus nullify all stage lowerings effected.

General Brown, then Chief of Engineers, United States Army, recognized the importance of reducing the height of floods by cutoffs, but felt that any channel improvements so accomplished should be maintained through adequate bank protection works. His attitude is clearly stated in an article[23] wherein he commented on the economic height of the levees that had been reached, and expressed the need for reducing flood heights as follows:

> "Our hope is to solve this problem satisfactorily. There are three phases of it that require careful consideration as well as physical research. They are: (1) Economical bank protection to stabilize the levee system and improve the channel conditions for navigation; (2) storage reservoirs; and (3) increased channel capacity by the elimination of bends. Of these three phases of the question, (1) and (3) are interdependent, (3) not being practicable unless (1) is solved."

The soundness of his logic has since been amply demonstrated, particularly that contained in the last sentence. Bank protection works have constituted an important phase of the cutoff project from the start. Under the direction of Maj.-Gen. Max C. Tyler, M. ASCE (then Brigadier-General), who succeeded General Ferguson in 1939 as President of the Mississippi River Commission, the cutoff work was extended farther upstream and considerable effort was devoted to developing better forms of bank protection. This effort was resumed under his successor, Maj.-Gen. R. W. Crawford, and today forms an important assignment receiving study at the Waterways Experiment Station (Vicksburg, Miss.).

In marked contrast with the objectives of Generals Brown and Ferguson in 1931 was the attitude toward any plan involving cutoffs on the part of the late Gen. T. H. Jackson, M. ASCE, then President of the Mississippi River Commission. In an article,[24] under the heading "Channel Shortening," General Jackson recited all the disadvantages of rectification by cutoffs, but he did not so much as mention the benefits of the stage lowering that could be derived from it, concluding with:

> "In flood-control plans of the past the cutoff has played an unimportant role, and there is little prospect of it playing a more important one in the future."

Colonel Vogel evidently labored under a misapprehension when he stated in his discussion that "General Jackson pioneered the way [for the Mississippi River cutoff program], overcoming prejudices and fears of long standing * * *"

[23] "Flood Control Work on the Mississippi," by Lytle Brown, *Scientific Monthly*, June, 1930, p. 499.

[24] "Flood Control on Alluvial Rivers with Particular Reference to Flood Control on the Mississippi River," by T. H. Jackson, *Engineering-News Record*, January 15, 1931, pp. 107–108.

and that General Ferguson "* * * applied the lessons developed by General Jackson's tests to the river as a whole." The writer can testify to the fact that General Ferguson, upon taking office at Vicksburg on June 15, 1932, found practically his entire staff strongly prejudiced against cutoffs, and, in addition, this prejudice had become extended to include many state officials and engineers of levee boards, as stated by Mr. Elam in his discussion. It is necessary also to correct certain misapprehensions regarding the cutoff experiments made at the Waterways Experiment Station in 1931. These had been ordered by General Jackson in a memorandum, dated January 2, 1931, in furtherance of a directive to him from General Brown, dated December 29, 1930.

These experiments have been referred to as helpful to the cutoff program. Aside from having demonstrated that cutoffs did not increase river stages downstream from them, the results obtained from the models were far from encouraging, and the summary of the report thereon was adverse to undertaking most of the cutoffs proposed, except for those at the Greenville Bends. Made in solid concrete models, and incapable of simulating bed and bank erosion, these experiments did little more than confirm what had previously been arrived at by that school of thought which clung to backwater computations and assumed that no changes in river cross sections would take place. In this respect these early experiments ignored the very issues that were basic to General Ferguson's plan of channel rectification and bank stabilization, particularly in regard to the importance of promoting bed erosion. Some use was made of sand and sawdust in these tests to determine where deposits were likely to form, but no thought appears to have been given to investigating the effect of lowering the river bed.

The tenor of the Waterways Experiment Station findings are best gaged by quoting from a report[25] as follows:

> "Item *e*: Any cutoff across Ashbrook Neck should be studiously avoided because of its probable effects on upstream velocities and downstream directions of currents. * * * Cutoffs at Caulk and Tarpley Necks can do little material good in reducing stages where desired, and although no objection can be seen to letting Caulk cut through by itself, there seems no valid reason for encouraging it.
>
> "Item *f*: The cutoffs between Vicksburg and Lake Lee would be valueless and extremely expensive to dredge. No further thought should be given to them. [This relates to Marshall, Willow and Sarah cutoffs.]
>
> "Item *g*: A cutoff at Diamond Point would have several good features and no apparent bad ones. Further tests will definitely indicate the advisability of assisting, or allowing it to occur naturally. As to the other three below Vicksburg: Yucatan is already accomplished; a cutoff at Natchez [later called Giles] would be of dubious value in an already fairly stable reach; and one at Esperance [later called Glasscock] of no value at all."

In the same report[26] the model of the Greenville Bends is stated to have produced a "Maximum lowering of 9.7 feet at mile 442.9." Then the head of the bends, as the result of simulating fully, developed cutoffs across Leland,

[25] "Experiment to Determine the Effects of Proposed Dredged Cut-Offs in the Mississippi River," *Report No. 1*, Waterways Experiment Station, Vicksburg, Miss., April 15, 1932, pp. 13–14.

[26] *Ibid.*, p. 40.

Tarpley, and Ashbrook necks. In contrast with this an observed lowering of more than 13 ft was actually realized at that point during the great flood of 1937, and this has since increased somewhat. Similar lowerings were registered also on the Arkansas City (Ark.) gage 6 miles farther upstream in 1937. This is illustrative of the marked increase in stage reduction effected wherever bed erosion and channel adjustment were made to play their parts, assisted where necessary by dredging.

Again, Caulk, Tarpley, Giles, and Glasscock cutoffs, contrary to model results, have proved effective. Glasscock, although greatly hampered in its early development due to the tenacious character of the soil which resisted erosion to a marked degree, has caused flood stages to become lowered all of 3 ft at Natchez, Miss., 18 miles upstream, corresponding to an increase in flood discharge of 200,000 cu ft per sec for rising river conditions. In regard to the Marshall and Willow cutoffs, which the report called valueless and extremely expensive to dredge and not worthy of further thought, it is only necessary to refer to the description in the paper of the case with which the pilot cuts were made (considering their lengths), and the satisfactory results obtained. Where the laboratory results showed a flood-stage lowering of a scant 3 ft at Vicksburg, General Ferguson brought about a lowering of 7 ft (which later increased to 10 ft) by lowering the bed profile along Racetrack Towhead by dredging.

The foregoing may indicate that these early experiments were of no help to General Ferguson, being in essence entirely at variance with the latter's conception of executing the work. It was not until after General Ferguson had taken office as President of the Commission, on June 15, 1932, that the Experiment Station received instructions to prepare river models with erodible beds. A series of studies of Mississippi River problems were undertaken that summer with erodible bed models.

In further reference to Yucatan Cutoff (an occurrence long considered inevitable), the writer's mention of its having been unexpected related to the fact that the break was caused by low-water caving at a time when no one was aware of it. Not an engineer, surveyman, or pilot of the Engineer Department was present to witness it. It was many days after the break that its occurrence became known to officialdom. The exact date of the occurrence of this remarkable natural avulsion has never been established.

It devolved upon General Brown to find a man to carry out the cutoff program together with all that was consistently and inseparably connected with it in the way of channel improvement and bank protection. Illuminating in this connection is General Brown's appraisement of the qualifications such a man should possess, as stated by him: "There are many who project an idea where danger is involved but there are few with the courage to give it effect and to assume the responsibility." He found a man who combined these qualities in General Ferguson, who not only had projected the idea, but possessed the indomitable nerve to tackle such a stupendous job and to see it through. General Ferguson stuck to his task more than seven years, until his retirement on August 15, 1939.

The location for a cutoff across Diamond Point, the first of the artificial cutoffs, was determined by General Ferguson on June 17, 1932, two days after

he assumed office. In the presence of the writer he drew the center line and the width of the right of way on a map, and dictated a directive "to have work initiated this summer with a view to acquiring right-of-way and completing a cut by dredging before next high water" (from the writer's personal diary dated June 17, 1932). A conference was held on that date in his office to discuss procedure. Those present were: Col. J. N. Hodges, District Engineer, New Orleans, La., in whose district the cutoff was to be made; his military assistant, the late Lt. J. P. Dean, United States Corps of Engineers; and Maj. P. S. Reinecke of the Commission staff. If any surveys or borings had been made previously, as mentioned by Colonel Vogel, these did not come to the writer's attention.

The writer is indebted to Dean Marston for presenting the work of the Mississippi River Engineering Board of Review on which he served. Credit is due the Board of Review for having given the practicability of the cutoff program very careful consideration. Its recommendations were in large measure responsible for the promptness with which authority for the work was granted by the Chief of Engineers. The Board of Review was appointed by General Brown by Special Orders No. 16, dated March 1, 1932, by order of the Secretary of War in furtherance of a resolution by the Committee on Flood Control of the House of Representatives, dated January 28, 1932, requesting that the Chief of Engineers

> "* * * examine and review the present status and condition of the works now in progress as authorized by the Flood Control Act of May 15, 1928 * * * with a view of determining if changes or modifications should be made in relation to the project and its final execution."

The Board of Review was directed by General Brown to determine, among other things,

> "* * * (d) the practicability and desirability of increasing the discharge capacity of the main leveed channel in flood time, and at the same time improving conditions for navigation. Measures for protection against flood waters that must of necessity escape from the main leveed channel of the river as modified by any conclusions as to channel enlargement."

These items had not previously been specifically authorized, except as covered under the broad head of channel stabilization. In addition to the three Board of Review members mentioned by Dean Marston, the Special Orders designated Lieutenant Dean to serve as "Recorder and Executive." Lieutenant Dean's services, due to his familiarity with the valley and its river problems, proved of inestimable assistance to the Board of Review on its inspection trips. The writer accompanied the board as civilian assistant to General Ferguson.

The Board of Review convened at Memphis (Tenn.) on March 9, 1932, at the office of the District Engineer, the then Maj. Brehon Somervell, Hon. M. ASCE, and almost immediately started on its first trip down the Mississippi River on board the steamer *Inspector*. The flood of that year was past the crest, and the river had dropped 20 ft, permitting an excellent view of bank and revetment conditions. In the lower valley, however, stages were still very high, and General Brown was at New Orleans to study the river while it was

at danger stage. The trip of the Board of Review extended down the main river to the mouth of Old River, thence through Old River down the Atchafalaya River. It included visits to two crevasses in the Atchafalaya Basin levees, the closure of which was in progress. After a visit to the Bonnet Carre spillway the trip ended at New Orleans on March 12, 1932. A second inspection trip, from May 8 to May 18, 1932, started from Cairo (Ill.) and was made partly by steamer and partly by automobile with much attention being given to problems in the St. Francis, Tensas, and Atchafalaya basins. The proposed enlargement of the Atchafalaya River to which Dean Marston referred was given considerable study on that occasion.

Answering Dean Marston's question regarding the development of the proposed enlargement as the result of the dredging operations performed, the Atchafalaya has been provided with two main channels emptying into Grand Lake by fairly direct routes. By 1940 these channels had become efficient outlets gradually replacing the previous labyrinth of bayous, thereby greatly improving flood-water outflow. The upper 56 miles of the original Atchafalaya River channel have been widened and deepened considerably through erosion by flood flow aided by supplementary dredging in places where tough bottom materials resisted erosion. The materials so scoured out in large part came to rest in the shallow waters of Grand Lake and are rapidly filling it. Two main outlets have been provided through these deposits—one by way of the old channel past Morgan City, La., and the other through a dredged channel into Wax Lake, and thence into the Gulf of Mexico. The discharge capacity of the Atchafalaya River proper has become greatly increased. During the flood of 1937 its maximum discharge confined between levees was in excess of 450,000 cu ft per sec. During the flood of 1945 discharge measurements showed further increases, the maximum flow being in excess of 600,000 cu ft per sec.

In regard to the question of leaving open the abandoned bend channels of the Mississippi River to serve as depositories for sediment and as high-water carriers and also for valley storage—these advantages are, of course, greatest and do the most good during the early development of a cutoff, but thereafter diminish in time. The resulting effects on the river are well described in the discussions by Mr. Pharr and by Messrs. Williams and Graves, and need not be restated, except to add that the Mississippi River is amply capable of transporting the entire load after the storage capacity of the bend ways become exhausted. The river is not a heavy sediment carrier compared to other western rivers; in fact, expressed in terms of parts per million of suspended load, it carries on an average only one tenth as much as the Missouri River.

Dean Marston's further requests for data are answered as follows: The planned flood-carrying capacity of the river between levees in the Arkansas City-Vicksburg reach now is 3,065,000 cu ft per sec. This is the maximum flood-crest flow estimated as possible of occurrence in that reach and was officially adopted as the revised project flow by the act of August 19, 1941. The capacity of 3,065,000 cu ft per sec is consistent with a similar maximum of 2,450,000 cu ft per sec provided for between Cairo and Arkansas City, and with a maximum of 3,000,000 cu ft per sec in the latitude of Angola (La.). The 3,065,000 value is appreciably larger than the 2,662,000 cu ft per sec, represent-

ing the estimated maximum confined flood-crest flow in 1927. The act of 1941 also provided for increases of freeboard on the main-line levees, which had become facilitated by the increases in discharge capacity. These increases made possible the further abandonment of the proposed floodways between the Arkansas River and the Red River. The 1941 standard levee requirements provided for a widening of the levee crowns to 25 ft to serve as roadways. Although much yardage remains to be placed to meet fully all these requirements, the height of the main-line levees in the Arkansas City-Vicksburg reach is now sufficient to take care of the new project flood flow.

In order to present as concisely as possible the picture of the development in channel capacity that has taken place since the first artificial cutoff was made in 1933, the approximate bankfull discharges at key points have been listed, year by year, in Table 7. The figures were estimated by inspection of the stage

TABLE 7.—INCREASES IN BANKFULL DISCHARGE CAPACITY AND REDUCTION IN FLOOD STAGE FOR RISING RIVER

Gage	Bankfull stage (ft)	Discharge Capacity (Million Cu Ft per Sec)								Total flood-stage lowering (ft)
		1932	1935	1937	1939	1943	1945	1946	Total increase	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Columbus, Ky.	43.0	....	1.00	1.03	....	1.10	1.10	....	0.10	1.5
Memphis, Tenn.	34.0	....	....	1.05	1.05	1.10	1.17	1.25	0.20	4
Helena, Ark.	41.0	....	0.94	0.96	0.98	1.00	1.10	1.20	0.26	5
Arkansas City, Ark.	44.0	1.00	1.12	1.65	....[a]	1.68	1.85	....[a]	0.85	15
Vicksburg, Miss.	42.0	1.00	1.17	1.30	1.38	1.45	1.55	....[a]	0.55	10
Natchez, Miss.	46.0	....	1.30	1.30	1.31	1.50	....	....[a]	0.20	3

[a] Flood crest did not reach top of banks.

discharge relations for rising flood stages only, as these afforded the only meaningful comparison. The point is that falling flood stages, because of the variability they exhibit in their stage discharge relations, are not useful for comparative purposes. Falling stages, moreover, are characterized by momentary changes in discharge, corresponding to changes in the rate at which the crossing bars are being built up, characteristic of rivers flowing in mobile beds. In interpreting Table 7 it is important to keep in mind that for falling bankfull stages the channel capacity is much less than shown, the reductions ranging from about 200,000 cu ft per sec to as much as 250,000 cu ft per sec, in some instances.

Statistically, flood-stage lowerings should be predicated on crest stages, but since no two floods are of the same magnitude in regard to crest discharge, a direct comparison is seldom feasible. In Table 8 an attempt has been made to compare the approximate increases in flood-crest stages that occurred in the past due to confinement by levees with the approximate flood-stage lowerings that have resulted since 1932 from the cutoff and channel improvement program.

The lowering of the river bed is in an entirely different category, being caused by scour, which in turn is affected by increased slope and lessened depth during floods—two factors which partly offset each other's effects. Since the bed of a meandering stream is a succession of deeps and shoals, it gives the low-

water profile a step-like appearance quite unlike the smoothly curving high-water profile. Available data show that the low-water lowering, although not as large in amount or as uniform as the high-water lowering, extends equally far upstream. Both eventually run out, and in each case the new over-all slope is steeper than the original slope before cutoffs were initiated. Table 5, supplied by Messrs. Williams and Graves, who have given this phase detailed

TABLE 8.—COMPARISON OF EARLY INCREASES WITH RECENT REDUCTIONS IN FLOOD-CREST STAGES

Increases in Flood-Crest Stages Due to Levee Confinement Prior to 1932				Reductions in flood-crest stages effected since 1932
Place	Year	Crest stage (ft)	Total increase (ft)	
(1)	(2)	(3)	(4)	(5)
New Orleans, La.	1849 1851 1922	15.2 15.4 21.27	6	Bonnet Carre spillway keeps stages below 20 ft
Baton Rouge, La.	1849 1850 1851 1927	34.93 34.53 34.53 47.80	13	Effect of Atchafalaya River enlargement and future use of floodways not known
Angola, La.	1851 1867 1927	46.4 46.3 57.5	11	
Natchez, Miss.	1770 1815 1927	48.0 48.6 56.65	8	3 ft
Vicksburg, Miss.	1828 1927	46.38 58.40	12	10 ft
Arkansas City, Ark.	1880[a] 1927	45.10 60.4	15	14 ft to 15 ft
Helena, Ark.	1850 1927	42.81 56.75	14	5 ft
Memphis, Tenn.	1828 1844 1850 1927	32.86 33.16 33.56 45.8	13	4 ft
Columbus, Ky.	1844 1858 1927	42.0 42.9 51.00	8 to 9	1½ ft

[a] No record prior to 1880.

study, shows the increases in low-water slopes that have resulted. Supplementing their data by reference to actual discharge measurements, it appears that the low-water plane for a uniform discharge of 150,000 cu ft per sec has been lowered as follows: Columbus, less than 1 ft; Memphis, about 3 ft; Helena, about 2¼ ft; Arkansas City, more than 3 ft; and Vicksburg, all of 10 ft.

As pointed out by Messrs. Williams and Graves, there has been no noticeable decrease in the low-water depths on the crossing bars since the opening of the cutoffs; what is more significant, low-water navigation has not been seriously inconvenienced. In fact, the depths on the crossing bars, throughout the length of river affected, have improved as the result of the work done, and in

1941 minimum depths of 12 ft were found to exist over most of the bars. At the few shallower crossings 12-ft depths were readily maintained by making cuts with dredges of the "dust-pan" type. In consequence, the previously prescribed 9-ft project depth between Cairo and Baton Rouge was authorized to be increased by an act of Congress on December 22, 1944, to 12 ft, leaving the 300-ft navigable width unaltered. Below Baton Rouge the old 35-ft project depth for sea-going vessels has remained in effect.

The continuity of the river-bed profile in the reaches north of Old River (to answer Dean Marston's inquiry) shows a decided improvement over that existing previous to 1933. The largest humps in the old profile have been eliminated, and the step-like succession of pools and crossings has become a gradually ascending series which possesses a considerable degree of regularity. An example of the nature of the adjustment that takes place is shown in Fig. 8. The new low-water profile shown by the heavy black line tends to approach, in the matter of spacing of pools and crossing shoals, the 5-mile interval, which in the early studies was found to be a characteristic of the river bed in its "orderly" reaches. A thin rubber band stretched over the 1946 profile in Fig. 8 will show that it averages a straight line, which is normal for a 25-mile distance.

The probable effect on the Lower Mississippi River of the flood control reservoirs in the Arkansas River watershed and on other Mississippi River tributaries is at present unpredictable. This is the writer's opinion, requested by Dean Marston. These reservoirs, however great their number eventually may become, are designed for the express purpose of providing protection to the lives and property in the valleys of the tributaries on which the dams are built, or are to be built. Operation by any coordinated plan still remains to be worked out, more particularly in regard to timing the release of the impounded floodwaters. The huge model of the Mississippi River watershed under construction by the Waterways Experiment Station at a site near Clinton, Miss., is designed specifically for shedding much-needed light on this complex problem. Furthermore, it is the opinion of the writer that the safety provided by the existing project for the Lower Mississippi River, although it may benefit incidentally from the operation of these reservoirs, is not dependent on such operation.

In answer to Colonel Hall's request for information concerning the economic aspects of the cutoff project, a segregation of costs would not be helpful because of the close interrelation of the several phases of the work. A statement follows which describes what the cutoff project has accomplished, what additional work it has entailed, and what additional improvements it has made possible that previously had remained unattainable.

River Channel.—The shortening by cutoffs and chute enlargements has eliminated excessive curvature in a river length of about 500 miles without straightening the channel unduly, or creating any tendency toward channel braiding. It has eliminated annual maintenance on 199.2 miles of abandoned bend channels, and has increased channel discharge capacity and reduced flood stages as shown in Table 7. The river bed has been lowered so as to leave low-water depths unreduced. It has made possible channel improvements by dredging which previously were not warranted economically. The amount of

dredging involved runs into large quantities but has produced permanent results essential, in most cases, to insuring the full measure of benefit derivable from the cutoffs. Further operations of this kind will reduce progressively as channel stabilization advances. The yardage excavated in the pilot cuts is a small item in relation to the yardage eroded by the river in the enlargement of the pilot cuts to full-sized river channels. Likewise, the yardage dredged for improving the river channel is a fraction of what the river has removed additionally.

Banks and Revetments.—Annual maintenance on 59.3 miles of existing revetments rendered inactive in abandoned bends has been largely eliminated. On the other hand, channel deepening upstream from cutoffs, in addition to the increased force of the current due to local profile steepening, has been responsible for local bank caving and the sliding and destruction of several revetments. Further trouble of this kind must be expected until hydraulic gradients become stabilized and the attack on the banks affected ceases. In this connection it seems pertinent to emphasize again the fact that as yet no adequate type of revetment has been devised for use on the Lower Mississippi River.

Control of Floods.—Major benefits have resulted from the lowering of flood stages. Only a meager conception of the magnitude of these lowerings is afforded by reference to Table 7. It is best conveyed by stating that they have made practicable of attainment the protection against the greatest flood estimated as likely to occur without having to raise the entire levee system. In the large backwater areas, where the most fertile soils are to be found, the reductions in the frequency, duration, and depths of flooding that have resulted have insured better returns from the land and have increased land values. Along the levees, the dangers from underseepage and landside boils have been lessened as flood heights have been reduced and the passage of flood crests has been expedited. In two respects no gains have been made—namely, the total length of the main-line levees has not become reduced by the channel shortenings; and the bank caving above newly made cutoffs or by readjustments in channel alinement will continue to require setting back levees, in places, until a less vulnerable type of revetment is evolved. Perhaps the greatest economic benefit that has been derived from the reduction of flood stages is the ability to dispense with a floodway on the west side of the valley between the Arkansas River and the Old River. This has removed the threat of inundation from many thousands of acres of valuable farming land and from large stands of hardwood timber. Finally, from the viewpoint of the citizens of Greenville, the removal of the river from their door by cutting off the Greenville Bends ranks as the outstanding benefit that has accrued to any community in the valley.

Navigation.—Distinct benefits have accrued to commercial navigation. Shortening the river has reduced the time of travel at ordinary stages appreciably, especially for ascending tows, despite the slight increase in current velocities due to slope steepening. Low-water depths are better than before cutoffs were made. Time lost at points where rapid channel changes were in progress has been a temporary inconvenience. Only during flood stages is there some increase in running time, but this is offset in part by the shortened duration of high flood stages. Thus summarized, a fair conclusion must be that the benefits by far outweigh the disadvantages, and, as a federal project, the investment is economically sound.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

CONSOLIDATION OF FINE-GRAINED SOILS BY DRAIN WELLS

Discussion

BY PHILIP KEENE

PHILIP KEENE,[24] ASSOC. M. ASCE.—A valuable contribution to the subject of drainage wells for hastening consolidation of fine-grained soils, a subject which is receiving increased attention in the field of soil mechanics, has been made by the author. After a study of the paper, the graphs become useful tools in analyzing actual problems and in designing installations, as mentioned by the author. Mr. Barron is also to be congratulated for his solutions showing the effects of smear and well resistance which were not presented before. When the wells are installed without casing or mandrel, the smear zone is probably less than ½ in. thick and can be ignored for wells larger than 6 in. in diameter. For other cases, the author's solution indicates that the smear zone materially affects the time factor and rate of settlement.

During 1947 the writer designed and supervised the installation of about 400 sand drainage wells on a highway project in Old Lyme, Conn.[25] A fill from 15 ft to 20 ft high was placed for a four-lane divided highway on a foundation of approximately 30 ft of soft organic clayey silt. Final settlements will average about 5.4 ft; 3.5 ft (or 65%) consolidation occurred during the six weeks necessary to place the fill. Assuming the average date of load application to be three weeks after commencement of placing the fill, 4.0 ft (or 75%) consolidation occurred five and one-half weeks after this average loading date. This is about 6% of the time estimated for 75% consolidation if no wells had been installed. Calculations made with the aid of Fig. 3 indicate that the actual time-settlement curve for that project will roughly approximate a theoretical time-settlement curve, if it is assumed that horizontal permeability is ten times as great as vertical permeability. Well resistance was not included in these calculations, although it was actually present in approximately the same value used in Fig. 10.

Correction for *Transactions:* In June, 1947, *Proceedings*, page 824, Fig. 10, the smear ratio s for cases A and B should read 1 rather than 0.

NOTE.—This paper by Reginald A. Barron was published in June, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1948, by Kenneth S. Lane.

[24] Associate Highway Engr. (Soils and Foundations), State Highway Dept., Hartford, Conn.

[25] "Sand Wells Speed Marsh Crossing," by Philip Keene, *Better Roads*, September, 1947.

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DISCUSSIONS

UNDERGROUND CONDUITS—AN APPRAISAL OF MODERN RESEARCH

Discussion

BY ZDENĚK BAŽANT, JR., ANSON MARSTON, AND GEORGE E. SHAFER

ZDENĚK BAŽANT, JR.,[37] ESQ.—In this interesting review of the test results obtained at the Iowa Engineering Experiment Station at Ames, the author states that rigid circular pipes of reinforced concrete are difficult to analyze by the principles of mechanics. The writer's opinion differs on this point, because he has used this type of analysis for the design of reinforced concrete pipes since 1944 with good results.

For ditch conduits the author uses a design load factor (under the heading, "The Supporting Strength of Underground Conduits") "* * * which is defined as the ratio of the strength of a pipe under any stated condition of loading to its three-edge bearing test strength." For concrete cradle (under the heading, "Ditch Conduit Beddings") he gives the load factor values varying from "2.2 to 3.4." The designer has a choice within these broad limits, but elsewhere he can find more precise data and can determine different load factors for various dimensions of cradles.[21]

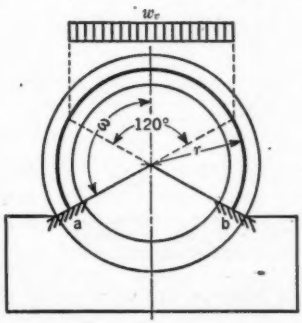

FIG. 20.—PIPE CONSIDERED AS CIRCULAR ARCH FIXED INTO CONCRETE CRADLE

If a form of cradle is designed for which values have not been published, the empirical load factor is not known; however, it can be calculated theoretically. The pipe is considered a circular arch fixed into the cradle. In the writer's opinion it is possible to use this method of calculation for all forms of cradles and to eliminate the empirical load factor completely. A comparison can be made of the accuracy of this assumption for the dimensions of the cradles tested.[21] As an example for comparison, pipes 24 in. in

NOTE.—This paper by M. G. Spangler was published in June, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1947, by Wilson V. Binger, and Jacob Feld.

[37] Associate Prof. of Foundations, Univ. of Technical Sciences, Prague, Czechoslovakia.

[21] "Concrete Cradles for Large Pipe Conduits," by W. J. Schlick and James W. Johnson, *Bulletin No. 80*, Iowa Eng. Experiment Station, Ames, Iowa, 1926.

diameter have been chosen. In the Iowa tests the pipes were tested with a 120° upper sand bearing, which causes uniform load. Theoretical representation of this load on a circular arch is shown in Fig. 20.

The maximum moment occurs at points a and b, where the first cracks occur and

$$M_a = M_b = M_o - H\,(r\cos\omega - t_o) - C = n_1\, w_v\, r^2 \ldots\ldots (16)$$

in which

$$M_o = -\,w_v\, r^2 \frac{\sqrt{3}}{2}\left(\sin\omega - \frac{\sqrt{3}}{4}\right) \ldots\ldots (17)$$

$$H = -\,w_v\, r^2 \frac{r\,\frac{\sqrt{3}}{2}\left(\sin^2\omega - \frac{\sqrt{3}}{2}\sin\omega + \frac{1}{4}\right) + t_o\left(\sqrt{3}\cos\omega + \frac{3}{4}\omega - \frac{5\pi}{12} - \frac{3\sqrt{3}}{8}\right)}{r^2\,\omega + r^2\sin\omega\cos\omega - 4\,t_o\, r\sin\omega + 2\,t_o^2\,\omega} \ldots (18)$$

$$t_o = \frac{r\sin\omega}{\omega} \ldots\ldots (19)$$

and

$$C = -\frac{w_v\, r^2}{2\,\omega}\left[\sqrt{3}\cos\omega - \frac{\sqrt{3}}{2}\left(\omega - \frac{\pi}{3}\right) + \frac{\pi}{6} - \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{2}\right] \ldots (20)$$

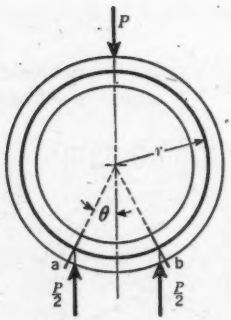

FIG. 21.—PIPE CONSIDERED AS CIRCULAR ARCH SUBJECTED TO THREE-EDGE BEARING

For three-edge bearing (Fig. 21),

$$M'_a = M'_b = \frac{P\,r}{2\,\pi}\left[\cos\theta - (\pi - \theta)\sin\theta + 1\right] = n_2\, P\, r \ldots\ldots (21)$$

From Eqs. 16 and 21, in which n_1 and n_2 are absolute numbers for definite ω and θ,

$$F_p = \frac{w_v\, r\,\sqrt{3}}{P} = \frac{\frac{M_a\,\sqrt{3}}{n_1\, r}}{\frac{M'_a}{n_2\, r}} \ldots\ldots (22a)$$

Because the pipe cracks occur in both cases under the same moment (that is, $M_a = M'_a$),

$$F_p = \frac{n_2\,\sqrt{3}}{n_1} \ldots\ldots (22b)$$

Numerical results are given in Table 1. The distance ab for three-edge bearing is assumed to be 3 in., thus making $\theta = 5°\,44'$. The load factor is expressed for ultimate load, from which the allowable load is obtained by applying a safety factor. The load factor computed for the first crack cannot be used for the comparison, because it is not proportional to the moments obtained by analysis. However, if Table 1 proves the usefulness of the writer's methods, simplifications are made by neglecting the normal thrust, the effect

of thickness of pipe, the influence of dead load of pipe, and the elastic fixing of pipe into the cradle.

Thus, the circular arch concept can be used for designing pipes on concrete cradles. Despite the elaborate appearance of the computation, the design is actually simple, because the n-coefficients can be calculated for each angle ω and for each type of load.

TABLE 1.—COMPARISON OF TEST RESULTS WITH THEORETICAL CALCULATIONS FOR 24-IN. DIAMETER PIPES

Type of cradle[a]	Thickness of upper part of cradle	ω	n_1[b]	n_2[b]	Load Factor, F_p	
					Observed for ultimate load	Calculated by Eq. 22*b*
(1)	(2)	(3)	(4)	(5)	(6)	(7)
4....	$\frac{b_c}{8}$	138° 35′	0.159	0.269	2.74	2.93
5....	$\frac{b_c}{4}$	120° 0′	0.131	0.269	3.25	3.56
8....	$\frac{b_c}{2}$	90° 0′	0.101	0.269	4.20	4.62

[a] "Concrete Cradles for Large Pipe Conduits," by W. J. Schlick and James W. Johnson, *Bulletin No. 80*, Iowa Eng. Experiment Station, Ames, Iowa, 1926, p. 39. [b] Absolute numbers, neglecting sign.

Security against the first crack is obtained with the writer's method by using a safety factor fixed experimentally for each kind of pipe so that tension stresses do not exceed the selected allowable value. Further investigation is needed to ascertain the exact manner in which the rigid cradle causes concentration of loads on the pipe. It is quite possible that it increases the load coefficient C. However, despite the present lack of data the writer considers this effect negligible.

ANSON MARSTON,[38] PAST-PRESIDENT AND HON. M. ASCE.—In this paper Professor Spangler has made a good, concise presentation of the writer's theory of external loads on conduits under fill materials. In the paragraph containing Eqs. 4, he quotes the writer as defining[4,6] "* * * the various settlements, y, as the * * * increments due to the addition at or above the height of equal settlement of any [the same for all] incremental uniform layer of fill materials." This definition is so worded because the calculation coefficient C_e in the load equation (Eq. 8) for projection conduits contains the value at height H_e of the coefficient C_e in Eq. 6*b*—thus providing for all fill loads (including their time lags) due to settlements (also including time lags) below H_e. To clarify its character further the plane at H_e in the writer's theory should be termed the "plane of equal additional settlements." It should be noted that all inequali-

[38] Dean Emeritus of Eng., Iowa State College, Ames, Iowa.

[4] "Second Progress Report to the Joint Concrete Culvert Pipe Committee," by Anson Marston, April 7, 1922 (mimeographed).

[6] "The Theory of External Loads on Closed Conduits in the Light of the Latest Experiments," by Anson Marston, *Bulletin No. 96*, Iowa Eng. Experiment Station, Ames, Iowa, 1930.

ties in the settlements of fill materials in the vertical prism over the culvert and those in the masses of "alongside" embankment materials, which develop during the construction work up to the height H_e, are leveled up in grading.

Following Eq. 8 in the paper, the author states two conditions that are used in deriving the writer's equation for evaluating H_e:

> "*a*. The internal friction in the embankment materials distributes the infinitely small increments or decrements of pressure from shear into the interior prism [that over the culvert] below the plane of equal [additional] settlement in such manner that their effect on settlement may be assumed to be substantially the same as for uniform vertical pressure; and
>
> "*b*. The internal friction in the embankment materials distributes the infinitely small decrements or increments of pressure from shear into the exterior prism so completely that their effect on settlement may be neglected" (but the "alongside" embankment fill materials are just masses of indefinite horizontal extent, not distinct vertical prisms of determinable—or assumable—width).

In the foregoing quotations the interpolations in parentheses are the writer's. Assumption *a* is the same as that previously found to be true (for ditch conduits) for all values of b_d investigated.[2,3] The late Milo S. Ketchum,[39] Hon. M. ASCE, verified it for distances of at least 40 ft between the walls of bins filled with granular materials. In each case, the horizontal distance over which uniform pressure due to shear is transmitted is that between two vertical planes, one on each side, along which slippage occurs because of settlement. In the cases of ditch conduits and culverts, the planes are those of the two sides of the vertical prism of fill materials overhead. For bins filled with granular materials the two planes of slippage are the side walls of the bins.

In reality, assumption *b* is the same as assumption *a*, but on each "alongside" there is no definite, opposite, vertical plane of settlement slippage in a typical embankment to stop the uniform distribution of pressure caused by the shear, due to the addition of the embankment materials above H_e before it has spread so widely as to have only negligible effect on settlement.

Concerning assumption *b*, the writer stated, in 1922,[4] in announcing the discovery of the "plane of equal additional settlement":

> "The only possible real test of the reliability of the new theory of loads on culverts from embankment materials (as proposed above) is comparison of the theoretical loads with the loads actually weighed in carefully conducted experiments with culverts"

—and he proceeded to show extremely close agreements of the final weighed and theoretical loads on the culverts under the 20-ft top soil (1919–1920) embankment and the 16-ft sand and gravel (1922) embankment. After 1922

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[2] "The Theory of Loads on Pipes in Ditches and Tests of Cement and Clay Drain Tile and Sewer Pipe," by A. Marston and A. O. Anderson, *Bulletin No. 31*, Iowa Eng. Experiment Station, Ames, Iowa, 1913, p. 32.

[3] "Loads on Pipe in Wide Ditches," by W. J. Schlick, *Bulletin No. 108*, Iowa Eng. Experiment Station, Ames, Iowa, 1932.

[39] "Design of Walls, Bins and Grain Elevators," by Milo S. Ketchum, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., 1919, pp. 310–311.

similar close agreement between weighed and theoretical loads has been found in nine more experiments at the Iowa Engineering Experiment Station (IEES) with different culverts under different embankments; and the correctness of the writer's theory has been so completely verified in addition by detailed field studies of more than five hundred culverts under highway and railway embankments, that it is entitled to be admitted unreservedly.

However, as stated by Professor Spangler in the text following Eq. 10, at least two possible modifications of the theory have been suggested. The most appealing of these is the one based on the assumption of a "plane of equal total settlements." This, as investigated by the author for the case of the IEES "permanent" concrete pipe culvert (constructed in 1927 and still under observation), gives different values for H_e for different values of H (beginning 5 ft too high as proved by the actual settlement measurements), and gives values for the culvert load 17% higher than that weighed at $H = 15$ ft and 11% higher than that computed for the writer's theory for $H = 73$ ft. This is not the method that Professor Spangler found (as stated, following Eq. 10) to give computed loads "* * * nearly the same as those obtained by Dean Marston's original analysis, being somewhat less for high values of $\frac{H_e}{b_e}$ and $\delta\,\rho$."

Any paper on the theory of culvert loads should include mention and discussion of the twelve culvert pipe load experiments made at the University of North Carolina at Chapel Hill, from 1924 to 1927, in two series.

First Series, 1924 to 1926.—Three tests were made with 30-in. cast-iron pipe sections (32 in. in outside diameter) projecting 50% above subgrade. In the first test, the embankment was of sand, 20 ft high; in the second test, the embankment was of clay, 14 ft high; and in the third it was of clay, 11 ft high, in which a ditch, 43 in. wide and 48 in. deep, was dug during construction directly over the pipe and then loosely refilled (copying a similar IEES experiment in 1920–1921 with a sand embankment). All embankments were kept covered by tarpaulins during construction "except during actual work on the fill or during pressure observations," and the clay embankments were rolled at each 1-ft layer above 3 ft, using a 4-ton roller. The first and second embankments were removed promptly on completion. The third was continued in place from February, 1925, to April, 1926. From August, 1925, the load decreased 33%, doubtless because of "* * * the drying and shrinking of the initially saturated clay, resulting from the summer's unusually low rainfall."[40] It then began a slow increase, which was still in progress when it was removed.

Second Series, 1926–1927.—In the second series of tests six 29-in. to 32-in. (inside diameter) pipe culverts (from 30.2 in. to 34 in. in outside diameters), and three 20-in. to 21-in. culverts (from 20 in. to 21.15 in. in outside diameters) were tested, successively (with 100% projection above subgrade), on the same load-weighing apparatus, under sand embankments 12 ft high above the culvert tops, which were kept covered by tarpaulins as for the first series. Each em-

[40] "Earth Pressure Experiments on Culvert Pipe," by G. M. Braune, William Cain, and H. F. Janda, *Public Roads*, November, 1929, p. 157.

bankment was removed promptly on completion. The culvert pipes were, respectively, of thin, smooth iron, corrugated metal, steel tubes, cast-iron pipe, reinforced concrete pipe, and solid concrete plugs.

The final detailed report[35] on these tests was made by the late G. M. Braune and H. F. Janda, Members, ASCE, with appendices by the late William Cain,[36] M. ASCE, long eminent for retaining wall pressure and column-formula mathematics. The support and weighing arrangements for the North Carolina tests were quite different from those adopted in the IEES experiments cited in Professor Spangler's paper.

The pipes were all mounted direct, without any bedding beneath, on what was practically like a solid rock formation. During the second series bank construction, some fill materials worked under; but a central strip from 60° to 80° wide was left bare.[36] The weighing apparatus was housed in a reinforced concrete structure on solid rock, covered by a reinforced concrete slab extending 10 ft, 7 in. on each side of the vertical prism over the pipe. This made the values of y_g and y_f each equal to 0 in the settlement ratio formula—

$$\delta = \frac{(y_m + y_g) - (y_c + y_f)}{y_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

so that

$$\delta = 1 - \frac{y_c}{y_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Because they did not recognize that the curved, lower parts of their plotted load curves proved the existence of planes of equal additional settlements at some heights H_e above the tops of the pipes, or that there were any settlement ratios δ, the North Carolina investigations made no measurements or discussions of y_m, or any other embankment settlements except y_c.

The use of tarpaulins and the prompt removal of embankments eliminated the time load lags, which other experiments and field studies prove would have added from 15% to 30% to the recorded culvert loads in the North Carolina tests.

The writer has made a thorough restudy of the detailed results of the North Carolina culvert tests, and finds that, adding proper percentages for time load lags, using the known value 1.00 for δ in the cases of the solid concrete plugs, and semi-empiric values of δ for the other culverts, the North Carolina culvert test results conform closely to results calculated by the writer's theory.

Neither Dean Braune[35,41] nor Professor Cain finally derived a definite comphehensive culvert load theory from the North Carolina results, although both toyed considerably with Dean Braune's idea[42] that they confirmed

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[35] "Earth Pressure Experiments on Culvert Pipe," by G. M. Braune, William Cain, and H. F. Janda, *Public Roads*, November, 1929, pp. 153–165.

[36] *Ibid.*, pp. 165–175.

[41] "Earth Pressures on Culvert Pipes," by G. M. Braune, *Public Roads*, January, 1927, p. 222.

[42] *Ibid.*, p. 226.

> "* * * the theory on culvert pipes, namely, that the pressures vary inversely as some power of the deflection of the pipe, and values of the ratio, K, have been obtained ranging between limits above and below unity."

A similar idea was stated in his final report. The "K" referred to was the ratio of the culvert load to the weight of the fill materials directly overhead. The idea that the culvert load might equal $K\ w\ b\ H$ was suggested by the writer in 1921 in his "First Progress Report," but was promptly abandoned by him in 1922, upon his discovery of the "plane of equal additional settlements."[44]

Professor Cain has stated[43]

> "All of the preceding results show that the deflection of the pipe is a leading factor in the theory of the vertical pressure actually exerted on the pipe"

and again[44]—"The value of K can only be obtained by experiment for a particular earth * * *."

GEORGE E. SHAFER,[45] ASSOC. M. ASCE.—This summary of valuable research on underground conduits should stimulate more practical use of the proved but too little accepted methods of determining the loads that drainage structures must withstand. For more than twenty years the writer has followed and helped disseminate the results of various research on culvert loads and design. Field experience during this period justified the feeling that much money has been wasted because engineers have not made use of the data available.

The writer had the pleasure of cooperating with Anson Marston, Past-President and Hon. M. ASCE, in the derivation and verification of the equations for "complete" and "incomplete" ditch condition discussed in the forepart of Professor Spangler's paper. Referring to the section, "Flexible Pipes," the writer and his associates were perhaps the first to recognize that flexible pipes do not fail as soon as the metal is stressed beyond the elastic limit, but continue to function structurally until the deflection resulted in a change in shape to the extent of reversing the curvature in the top or bottom of the structure. If one agrees with the author that structural failure of corrugated metal structures is due to excessive deflection, the problem then is to determine a method of predicting deflection.

Referring to Fig. 22, a flexible pipe is in equilibrium when the inherent strength of the pipe, R_I, plus the side support, W_H (developed in the earth as passive pressure due to outward movement of the sides of the pipe), is equal to the vertical load, W_c, on the pipe and its accompanying reaction.

How much a pipe deflects depends on all three factors—the larger the values of R_I and W_H the smaller will be the vertical deflection, y_c, and the larger the

[43] "Earth Pressure Experiments on Culvert Pipe," by G. M. Braune, William Cain, and H. F. Janda, *Public Roads*, November, 1929, p. 169.

[44] *Ibid.*, p. 175.

[45] Chf. Engr., Armco Drainage and Metal Products, Inc., Middletown, Ohio.

value of W_c, the greater that deflection. Therefore,

$$y_c = f\left(\frac{W_c}{R_I + W_H}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25)$$

Eq. 15 is in this form and includes those three factors.

Prior research on the design of flexible structures resulted in an empirical equation for deflection. This research should be considered, so that the results of the two equations can be compared. In 1926, after it was fully recognized that flexible structures fail by change in shape (excessive deflection), the writer and his associates set out to determine: (1) The law of deflection, so that structural performance could be predicted, and (2) the maximum permissible deflection, so that some part of the maximum could be used as a safe design deflection.

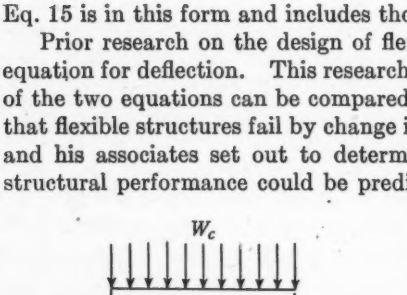

FIG. 22.—CONDUIT LOADING DIAGRAM.

As to the law of deflections, it was known that deflection would vary directly as some power of the fill height, H, and the diameter, D, and inversely as the thickness of metal, t. Thus,

$$y_c = k\frac{H^m D^n}{t^s} \quad \ldots\ldots\ldots\ldots (26)$$

Using the deflection data from the Ameican Railway Engineering Association(A.R.E.A.) investigation,[46] wherein the deflections were accurately measured as the fill was slowly made (permitting partial settlement to take place), along with measurements taken on numerous large diameter, high fill, unstrutted structures under both highway and railroad service, the values of k, m, n, and s were determined.

The maximum deflection before failure occurred was readily determined by inspecting numerous, large diameter pipe installations with shallow cover of the "horse and buggy" days in California. With better roads and flatter grades, many of these structures, because of their good material and structural performance, were left in place and extended to permit the fill height to be increased. In some cases, the additional cover was sufficient to cause considerable additional deflection. The maximum recorded was 22% at a localized point. The average safe maximum deflection was determined as 20% of the vertical diameter. Using the conservative factor of safety of 4, the design deflection was established at 5%.

During the thirty years prior to 1926, when the empirical equation for riveted corrugated pipe was derived, the use of corrugated pipe was steadily increasing. By 1926 there was in use approximately 100,000,000 ft of average size corrugated pipe. With acceptance and use, gage tables based on experi-

[46] "Culvert Load Determination," *Bulletin No. 284*, American Railway Engineering Assn., 1926.

ence had been compiled. Each large producer had his own gage table; however, all were practically the same. A new gage table based on the empirical equation using 5% deflection checked very closely with the old gage tables then in use, which were based on experience. Both were based on average installation conditions.

At that time, especially in certain parts of the United States, there were some questions regarding the durability of light gage, metal drainage structures. Gages that satisfied structural requirements often had to be increased because of durability requirements. For this reason, a gage table based solely on the equation was impractical. One of the objectives of the A.R.E.A. investigation[46] was a rational design for flexible pipe. The objective was never obtained; the empirical equation likewise fell short of the A.R.E.A. objective. It was also realized that the empirical equation was new and somewhat radical in principle. For these reasons it was not published. Like a new and untried tool it was used with discretion until its merit was established.

With the more general use of asphalt coated and paved invert pipe, plus the introduction of asbestos bonded sheets to insure adhesion of the asphalt to the metal, it was not so necessary to increase the gage of metal over and above the structural requirement. This permitted a more rigid use of the empirical equation. By 1931, when a new type of large diameter metal pipe made by bolting corrugated curve structural plates together in the field was introduced, there was sufficient confidence in the original empirical equation for riveted pipe with corrugations of $2\frac{2}{3}$ in. by $\frac{1}{2}$ in. so that a new equation was derived from field measurements of unstrutted structural plate pipe as soon as sufficient structures were in service to represent a good average. The equation was later revised as more installations were made and a new equation was derived for field strutted structures. These equations served to predict deflection. The bolted longitudinal seam was designed separately. Thousands of riveted and bolted structures have been designed on the basis of the empirical equations with almost faultless structural performance.

None of the equations have been published—not because they were considered trade secrets, but because the men who derived them knew their limitations. The equations have just one application—to determine the required thickness of metal for a given diameter of pipe and height of cover or dead load. A tabular form giving the required gage for all structurally safe combinations of diameter and fill height is a gage table. Such tables, based on all three equations, have been available since the equations were derived. In some cases, light gages have been modified because of durability requirements and in others, to take care of various live load conditions. Since a gage table is far more convenient to use than an equation, the public or profession has not suffered because the exact equations were not known. The form of the equation and the fact that the gage tables were based on an empirical equation were published in 1937.[47]

When Professor Spangler proposed his investigation on the structural design of corrugated pipe various members of the industry took interest and cooper-

[47] "Handbook of Culvert and Drainage Practice," Armco Culvert Mfrs. Assn., Middletown, Ohio, 1937.

ated. Those familiar with the empirical design hoped a rational design that would take into account installation conditions, type of soil, and other factors would result, so that, when possible, a more economical design could be used. When the investigation was completed and Eq. 15 was published, its use was investigated. The apparent weakness of the equation at that time necessitated the selection of the proper value of ϵ (modulus of passive resistance of the soil) to use for various soils. Later, the writer tried to use the equation to verify some live load tests and extend them into a gage table for airport loading. Again it was difficult to find the proper value of ϵ.

Perhaps the best way to discuss the application of Eq. 15 is to compare the permissible heights of cover for a range of diameters and gages with those obtained by the empirical equation. In Table 2, the heights of cover for Eq. 15 are based on $\Delta = 0.1\ r$, or 5% deflection, with $F_d = 1.5$, $F_k = 0.1$, E = 30,000,000, and $\epsilon = 20$. In general, the two equations agree fairly well for certain combinations of diameter and gage, but vary widely for other combinations. Eq. 15 does not give as much value to difference in gage as does the empirical equation and as is warranted by experience. It also permits higher fills for large diameters and light gages.

TABLE 2.—COMPARISON OF FILL HEIGHTS, IN FEET, OBTAINED BY EMPIRICAL EQUATION AND BY EQ. 15

Gage	24 IN. IN DIAMETER		48 IN. IN DIAMETER		72 IN. IN DIAMETER	
	Empirical	Eq. 15	Empirical	Eq. 15	Empirical	Eq. 15
16	26	24.2	7.2	15.9	3.5	21.7
12	62	36.6	18	17.4	8.8	22.2
8	140	53.7	39	19.6	18.5	22.2

Those engaged in the manufacture and distribution of flexible drainage structures see a definite need for a rational method of design. Not because design based on experience, such as the empirical equation, is wrong, but because it is possible that the full economy of flexible construction is not utilized in all cases. Furthermore, engineers prefer a rational approach to any problem, even though they use empirical methods in the solution of many problems. The rational approach also leads to a clearer understanding of the basic principles involved.

With this desire for a rational design in mind, the writer makes the following comments on each factor in Eq. 15 in the hope that by constructive criticism a solution will be evolved that will form the basis of all flexible conduit design.

Referring back to Eq. 25, the simplified form of Eq. 15, the writer has already implied agreement with the Marston theory for determining the vertical load on flexible structures. The part of Eq. 15 involving inherent strength R_I is based on the elastic theory of a closed ring. Referring to Fig. 23, this means that the load-deflection relation is a straight line and may coincide with the actual deflection curve from two-edge bearing tests for small deflections, or may deviate appreciably when the ring is stressed beyond the elastic limit.

Regardless of whether the load for two-edge bearing is determined from the equation,[30]

$$y_c = 0.149 \frac{P r^3}{E I} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

or for the factor of Eq. 15 corresponding to R_I in Eq. 25, the result is the same, when F_d and F_k are assumed equal to 1.5 and 0.1, respectively. Therefore, the strength that is built into the pipe can be determined by Eq. 27. It is realized that Eq. 27 does not apply beyond the elastic limit of the metal, yet the comparison is made on the basis of 5% deflection, slightly beyond the limit, because that deflection has become standard practice in the design of corrugated metal pipe. In Table 3, values of P or W are given for 5% deflection and for a representative range of diameters and gages. The data on two-edge bearing loads does not include the Iowa tests, but does include unpublished data from several other investigations.

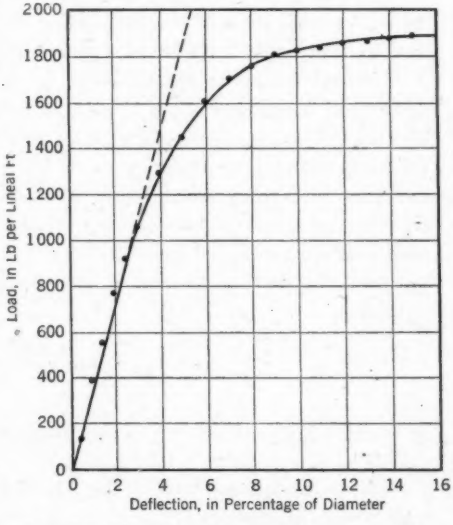

FIG. 23.—LOAD-DEFLECTION CURVE FOR 36-IN. DIAMETER CORRUGATED METAL PIPE (WALL THICKNESS, 0.112 IN.; ½-IN. CORRUGATIONS)

The loads determined by Eq. 27 for pipe 24 in. in diameter are approximately 2¼ times those from tests, and for pipe 72 in. in diameter they are approximately equal. If the effect of gage is considered regardless of diameter, Eq. 27 shows 8-gage pipe to be approximately 2¾ times stronger than 16-gage pipe, whereas the test data show this ratio to be approximately 4. These differences tend to affect the final design resulting from Eq. 15 by failing to show the full effect due to gage, regardless of diameter. This is a general comment regarding the use

TABLE 3.—COMPARISON OF INHERENT PIPE STRENGTHS[a] AS DETERMINED BY EQ. 27 AND BY ACTUAL TESTS

Gage	24 IN. IN DIAMETER		48 IN. IN DIAMETER		72 IN. IN DIAMETER	
	Iowa[b]	Test[c]	Iowa	Test	Iowa	Test
16	3,530	1,300	880	500	375	295
12	6,080	2,826	1,510	1,110	675	665
8	9,630	5,220	2,370	2,100	1,060	1,190

[a] Load in pounds per foot at 5% deflection. [b] Calculated by Eq. 27. [c] Determined by actual tests.

[30] "The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Iowa Eng. Experiment Station, Ames, Iowa, 1941.

of Eq. 15 in actual practice. There are two other factors that should be considered with the foregoing data: (1) How much stronger is a flexible pipe in uniform vertical loading, as compared with concentrated loading, and (2) how much does uniform loading raise the elastic limit over that for concentrated loading?

The usually accepted conversion factor for uniform loading from two-edge or three-edge bearing for rigid pipe is 1.5. According to the elastic theory, either by deflection of the ring or for moment in the ring, the factor should be approximately 2.0. It is the writer's opinion that a ratio of 2.0 is conservative.

Undoubtedly the elastic limit is higher (Fig. 23) for uniform loading as compared to concentrated loading, because of the more gradual and uniform change in shape or curvature under uniform load. Just how much the limit is raised is difficult to determine. If the change is small, then factors in Eq. 15 entering into inherent strength should be changed so that instead of being based on the elastic theory they would be based on actual concentrated load test data corrected for uniform vertical load. The average concentrated loads per foot of pipe to produce 5% deflection can be determined from the empirical equation,

$$P = \frac{4.825 \times 10^6\, t^{1.40}}{D^{1.35}} \qquad (28)$$

in which t is the thickness of the metal in inches (corrugations $2\frac{2}{3}$ in. by $\frac{1}{2}$ in.) and D is the pipe diameter in inches.

The side support part of Eq. 15 is based on certain assumptions, two of which will be discussed:

1. " * * * the horizontal pressure on the (side of a) pipe at any point bears a nearly constant relationship to the horizontal movement of the point."[30]

2. The distribution of passive pressure on the sides of the pipe is assumed symmetrical about the horizontal axis.

Considering first the assumption that horizontal pressure has a constant relationship to movement of the pipe sides, it is true that the Iowa data support this contention. Earlier research work[35] at the University of North Carolina (Chapel Hill) seems to offer verification, except for very low fills. Referring to Fig. 24, the assumption is correct when the pressure and movement curves are straight lines. The pressure curves are straight lines, but the deflection is not a straight line for low fill heights. The deflection curve for Professor Spangler's first experiment has a similar characteristic.

If the pressure movement ratio is constant, the value of ϵ should be a constant for any soil characteristic and should not vary with fill height, or with the degree of rigidity of the pipe. There seems to be evidence that ϵ varies with fill height, or pipe rigidity—or both. Considering first the data supporting the claim that ϵ varies with fill height, in Fig. 24 the value of ϵ calculated using Eq. 15 varies from less than 30 for a 1-ft fill to more than 70 for a 12-ft fill. Similar values of ϵ for the 20-in. pipe in the same investigation are not as con-

sistent as those plotted in Fig. 24 for the 30-in. pipe, but do vary from 28 for a 2-ft cover to 56 for an 8-ft cover.

In 1928 accurate deflection measurements were taken on a rather large number of metal pipes installed in a new line of the Union Pacific Railroad

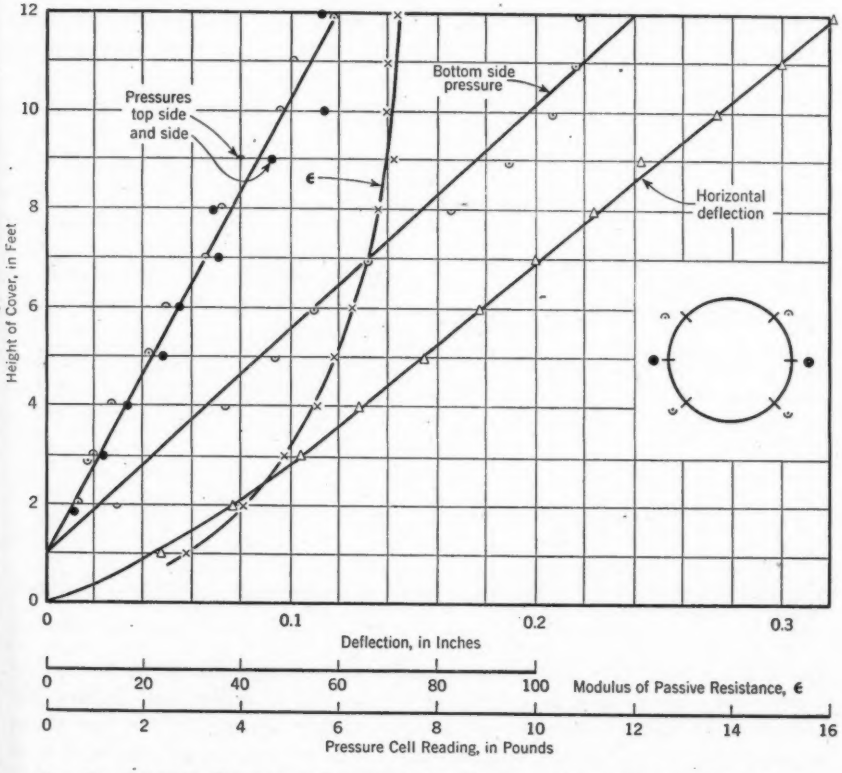
Pressures top side and side
Bottom side pressure
Horizontal deflection
Height of Cover, in Feet
Deflection, in Inches
Modulus of Passive Resistance, ϵ
Pressure Cell Reading, in Pounds

FIG. 24.—PRESSURE DEFLECTION DATA FROM UNIVERSITY OF NORTH CAROLINA STUDIES ON 30-IN. DIAMETER CORRUGATED METAL PIPE (WALL THICKNESS 0.108 IN.)

Company. Original diameters were measured before backfilling, and final readings were taken nine years later. The number of unstrutted structures for which there was complete data (diameter, gage, average load, two-edge bearing strength at 5% deflection, and calculated ϵ according to Eq. 15) are shown in Table 4. Also included in Table 4 are similar data on two other investigations—those at the University of North Carolina[30] and the unpublished data on a live load test made by the writer. As further proof that ϵ may vary with the fill height, note how the average ϵ decreased with the average fill. The thirty-eight structures included in this investigation were installed under various conditions. For that reason, a comparison between individual structures is unwarranted, although a comparison between the averages for any pipe size seems permissible.

Referring to Table 4(*b*), the North Carolina tests were carefully made so that all pipes were installed and backfilled as nearly alike as possible. The three pipes listed are those that can be considered as flexible, the most flexible being a smooth pipe that had to be braced to hold the circular shape until the

TABLE 4.—COMPUTED VALUES OF ϵ FROM VARIOUS TEST PROGRAMS

Number of structures	Diameter (in.)	Gage	Average Load		Bearing strength of pipe[a] (lb)	Computed ϵ	Col.6 / Col.7
			Dead (ft of fill)	Live			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(*a*) UNION PACIFIC RAILROAD COMPANY							
3	24	10	34	E60	3,950	62	64
16	36	10	17	E60	2,300	38	61
19	48	8	5	E60	2,100	18	117
(*b*) UNIVERSITY OF NORTH CAROLINA							
1	30[b]	12	12	0	....[c]	29	....
1	30	12	12	0	1,620	47	36.3
1	30[d]	$\frac{1}{16}$ in.	12	0	....[c]	89	....
(*c*) LIVE LOAD							
1	24	14	1	Truck with dual tires	1,780	262[e]	6.8
1	24	12	1		2,820	339[e]	8.3
1	48	12	1		1,110	149[e]	7.5
1	48	10	1		1,580	188[e]	8.4

[a] Two-edge bearing strength at 5% deflection. [b] Smooth pipe, very flexible. [c] Unknown. [d] Steel tube.
[e] Average for three different live loads (4,635 lb, 8,700 lb, and 13,695 lb) per wheel.

pipe was covered with sharp bank sand. The second and third pipes were progressively more rigid, as evidenced by the fact that the load factors (the percentage of load directly above the pipe actually carried by the pipe under 12-ft fill) were 61, 80, and 87, respectively. The values of ϵ, as calculated from Eq. 15 are 29, 45, and 85, respectively. These data tend to show that ϵ is a function of pipe rigidity. In Table 4(*c*) live load is the predominant load. All structures were carefully installed under the same installation conditions and loaded in the same manner. In this case ϵ varies with the pipe rigidity. It will be noted that the ratio of two-edge bearing strength at 5% deflection to ϵ is much more constant than the values of ϵ.

To summarize the data in Table 4, there is evidence to show that ϵ varies with pipe rigidity more than with fill height. Nevertheless, in the Union Pacific data (Table 4(*a*)) for 36-in., 10-gage pipe and the 48-in., 8-gage pipe, the rigidity is approximately the same, where average ϵ varies from 38 for an average fill of 17 ft to 18 for a fill of 5 ft. This, added to the variation in ϵ shown in Fig. 24 for the same pipe under varying fill heights, leads the writer to believe that ϵ may vary both with fill height and pipe rigidity.

It is unfortunate that all the pipes in the author's third experiment, the major part of his tests wherein conditions were identical, were of approximately

the same rigidity. In selecting the size and gage of pipe to use, the aim was to keep them as flexible as possible within the limits of practicability. The inherent or two-edge bearing strengths of the four pipes in the third experiment, according to the writer's data, are shown in Table 5. Had the tested pipe been of widely different degrees of rigidity, the resulting data might have indicated that ϵ is a function of pipe rigidity.

TABLE 5.—TWO-EDGE BEARING LOAD FOR PIPES OF THE AUTHOR'S THIRD EXPERIMENT

Diameter (in.) (1)	Gage (2)	Two-edge bearing load[a] (lb) (3)
36	16	750
42	14	840
48	14	685
60	12	815

[a] At 5% deflection.

Considering the second assumption, that the passive pressure on the sides of the pipe is symmetrical about the horizontal axis, the right half of Fig. 25 shows the development by Professor Spangler of a "witch-shaped" (witch of Agnesi) curve representing the locus of the horizontal movement of all points on the half circle, as the circle changes to an ellipse, their horizontal axes being coincident. In actual installations the inverts are coincident if there is no settlement. The left side of Fig. 25 shows the approximate shape of the locus based on coincident inverts and the same parabolic curve Professor Spangler

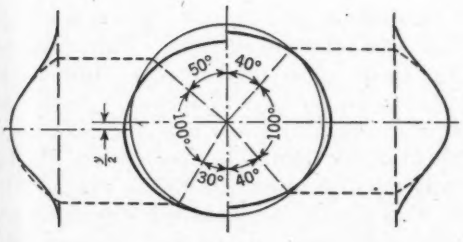

FIG. 25

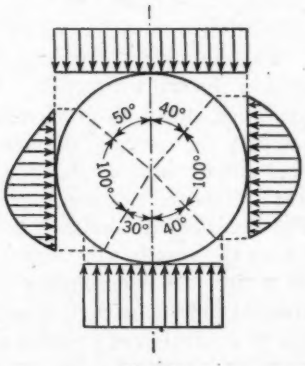

FIG. 26

used, which closely simulates the witch curve. The new center of the parabola is below the horizontal axis of the pipe by an amount equal to $y/2$. Actually, the difference in shape of the sides is not great, but the difference in shape combined with increased pressure below the horizontal could appreciably change the pressure diagram or composite loading.

Referring to the North Carolina data[35] for the 30-in. pipe, the pressure curves in Fig. 24, which were drawn to show the increase in pressure starting at a 1-ft fill, show the average pressure 45° above the horizontal "top side" to be approximately equal to the average side pressure. The average pressure 45° below the horizontal "bottom side" is more than twice the side pressure. This does not mean that the horizontal component of the measured pressure on the bottom side is greater than the horizontal component on the top side. It does,

however, indicate that there can be a high pressure under the haunches of the pipe. Professor Spangler's composite loading, on which Eq. 15 is based, is shown on the right side of Fig. 26. The pressure at a point 45° below the horizontal, according to this loading, is very small. The data justified that assumption, but the writer believes that, had the pipe in the tests deflected more, or had the pipe been installed under 100% projection, the load diagram would be similar to that shown on the left side.

As further proof of greater pressure under the haunches than that assumed by the author, the averages of all the pressure cell readings for the three flexible pipes in the 30-in. size tested at North Carolina[35] are shown in Fig. 27. These pressures are in pounds per square inch, and are the average increase in pressure due to any additional foot of fill from 0 ft to 12 ft. These data show that, with the exception of the top point, the greatest pressure is at the haunches, or 45° below the sides. Previously published data[48] show the haunches to be points of low pressure for low fills, with a trend toward more uniform pressures for higher fills. Such a revised pressure diagram would complicate the mathematical analysis but would, in the writer's opinion, more nearly fit field conditions for design deflections and the usual installation conditions, wherein the foundation is often quite yielding.

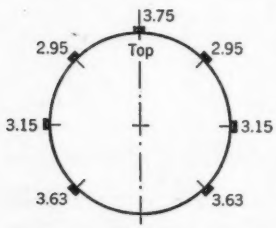

FIG. 27

To summarize, Professor Spangler's investigation and interpretation are steps in the right direction. It is unfortunate that the test pipes in the third experiment were all of virtually the same inherent strength. This, plus the fact that the range of diameters was small, apparently led to a solution that gives undue value to the side support factor for large diameters. Additional data are needed on working values of ϵ, especially under live load conditions. Whether there are sufficient test data available from all sources to revise Eq. 15, so that the results will be more consistent with actual practice, is unknown to the writer. It is emphasized again that the comments in this discussion are intended to be constructive and to encourage ideas and data that will eventually lead to a solution of a rather complex problem.

[48] "The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Iowa Eng. Experiment Station, Ames, Iowa, 1941, Figs. 24 and 25.

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DISCUSSIONS

THEORY OF INELASTIC BENDING WITH REFERENCE TO LIMIT DESIGN

Discussion

BY E. P. POPOV, AND BERNARD A. VALLERGA

E. P. POPOV,[12] ASSOC. M. ASCE—A timely study of the theory of inelastic bending, with particular reference to mild steel, is presented in this paper by Professor Hrennikoff. Engineers engaged in the aircraft industry are re-examining the usual criteria of design based on stress and, in certain cases, are attempting to substitute in their place the criteria of excessive deflections, or the ultimate carrying capacity of the flexural members. Historically, it is of interest to note that the early elasticians were not satisfied with the direct proportionality of stress to strain dictated by the Hooke-Mariotte law. B. de Saint Venant,[13] in his notes to M. Navier's book, thus considers a generalized stress-strain relation and obtains the resulting stress distribution. In recent years most engineers are familiar with the assumption of parabolic stress distribution in concrete.[14]

With the demand for rapid analytical solutions, which required simple formulas, straight-line stress distribution became universally adopted and stress became the criterion of design. This probably also followed from researches on fatigue, where stresses and stress concentrations play the major role in determining the serviceability of a member. Gradually, it became apparent that in a large number of cases reversal of loads does not occur. Hence, it is argued, stress is not the best criterion for the selection of a member. This explains the renewed activity in researches in plastic bending, and attempts are made to determine the ultimate carrying capacity of members (limit design).

All problems of plastic bending in earlier and present analyses are based essentially on the Bernoulli-Euler assumption that plane sections, before bending, remain plane after bending. Hence, it follows that strains vary lin-

NOTE.—This paper by Alexander Hrennikoff was published in March, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1947, by George Winter, and D. C. Drucker; and October, 1947, by Michael R. Horne.

[12] Asst. Prof., Civ. Eng., Univ. of California, Berkeley, Calif.

[13] "A History of the Elasticity and Strength of Materials," by I. Todhunter and K. Pearson, Cambridge Univ. Press, Cambridge, England, 1893, Vol. II, Pt. I, pp. 115–119.

[14] "Reinforced Concrete Construction," by G. A. Hool, McGraw-Hill Book Co., Inc., New York, N. Y., 1927, Vol. I, p. 60.

early from the neutral axis. This assumption has a reasonable experimental support, as shown by A. Nadai,[15] based on experiments of C. V. Bach, E. Meyer, G. H. McCullough,[16] Joseph Marin, Assoc. M. ASCE, R. L. Cotterman,[17] Jun. ASCE, and others. With this basic assumption the solutions obtained and reported differ merely in form. Professor Hrennikoff chooses certain nomenclature and derives certain expressions, which, in turn, are evaluated for a particular stress-strain diagram.

In 1943 F. P. Cozzone[18] derived formulas and graphs for the plastic-range design of several materials. In his presentation the stress-strain diagram is treated as an equivalent trapezoid, having the same moment about the vertical axis as the actual stress-strain diagram. Earlier, S. Timoshenko,[19] using the conjugate beam method, and utilizing the notion of reduced modulus of elasticity, devised a semigraphical method of determining stresses and deflections. All these schemes are identical in principle and differ only in form. The solutions, obtainable by the use of these methods, should be in substantial agreement. For pure flexure, or for cases where shear is small, it appears that these solutions are in good agreement with previous tests[17,18,20] and may be used with confidence.

To simplify his analytical treatment for I-beams and WF-beams Professor Hrennikoff also introduces a ratio κ, which is defined as a ratio of flange area to one half the area of the web. This simplification also appears reasonable, but the situation differs with shear.

The writer seriously questions whether the simple procedure developed in the paper for shearing stresses in beams is in agreement with known facts about the plastic behavior of ductile materials. Shearing stress concentrations show in Fig. 17 and, again, shearing stress distribution, shown by the dotted line in Fig. 18 of the paper, are excessively high and appear to be strangely localized. Referring to Mr. Cozzone's paper,[18] in which the same basic assumption is utilized, the shearing stress distribution is shown to be

$$\tau = \frac{V\,Q}{I\,b}\left(\frac{1 + R\left(\frac{A\,C}{Q} - 1\right)}{1 + R\,(k - 1)}\right) \qquad (56)$$

In Eq. 56 τ, V, b, and I have the same meanings as in the paper; A is the area between the fiber in question and the extreme fiber; Q is the first moment of A about the neutral axis; c is the distance to the extreme fiber; k is the section form factor ($1 \leq k \leq 2$); and

$$R = \frac{df_0}{df_m} \qquad (57)$$

[15] "Plasticity," by A. Nadai, McGraw-Hill Book Co., Inc., New York, N. Y. 1931, p. 160.

[16] "An Experimental and Analytical Investigation of Creep in Bending," by G. H. McCullough, *Transactions*, A. S. M. E., 1933, p. 55.

[17] "Experiments on Plastic Bending for Aluminum Alloy 17S-T," by J. Marin and F. D. Cotterman, *Proceedings*, A. S. T. M., Vol. 43, 1943, pp. 581–599.

[18] "Bending Strength in the Plastic Range," by F. P. Cozzone, *Journal of The Aeronautical Sciences*, May, 1943, pp. 137–151.

[19] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., New York, N. Y., Pt. II, 1941, pp. 362–373.

[20] "The Bending Modulus of Cast and Forged Sections," by N. L. Willett and L. Bonza, *Lockheed Report No. 2772*, Burbank, Calif., 1942.

which is the rate of change of intercept stress f_0 with respect to the maximum stress f_m. These latter quantities require some additional comment.

In the Cozzone analysis, in order to avoid algebraic difficulties, the actual stress-strain diagram that corresponds to any maximum stress f_m in a beam is replaced by a trapezoid. The ordinary tension test diagram is used as a basis. Let Oab in Fig. 36 represent such an actual tension test diagram. Then, for a particular moment that a beam carries, if the maximum stress is found to be f_m in the usual way, the stress distribution in that beam will be similar to area

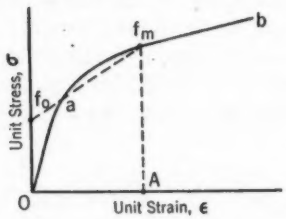

FIG. 36

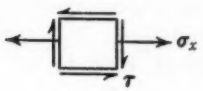

FIG. 37

Oaf_m. The moment of the area Oaf_mA about the vertical axis through the origin is proportional to the beam's flexural capacity at stress f_m. To simplify further derivations, a trapezoid Of_0f_mA is then found such, that its moment around the vertical axis through point O will be equal to the moment previously found. Thus, for a particular value of f_m there is a particular corresponding value of stress intercept f_0. As the applied moment to the beam is increased, f_m increases and f_0 changes correspondingly.

In Eq. 56, R is defined as the rate of change of intercept stress f_0 with respect to the maximum stress f_m. During the plastic stage (that is in the interval between ϵ_{Y1} and ϵ_{Y2} in Fig. 13), f_m remains stationary, whereas f_0 may vary slightly. Thus, Eq. 56 reduces to

$$\tau = \frac{V\,Q}{I\,b} \qquad (58)$$

which is a well-known expression for the shearing stress distribution in the elastic range developed by D. I. Jouravsky[21] in 1855.

Thus, the analyses of Professor Hrennikoff and Mr. Cozzone in the plastic range lead to very different results. It is the writer's opinion that in the plastic range problems concerned with shear cannot be handled in such an elementary way. The flow criterion for ductile materials, such as the steel considered in Fig. 13, is expressed well by the Hencky-Mises-Huber relation.[22] In it the shear distortion energy under a compound state of stress is equated to the shear distortion energy in simple tension at the yield point of the material, σ_{yp}. In bending, with shear, the state of stress on an element in a beam is represented in Fig. 37, and the plastic flow criterion becomes:

$$\sigma_{yp} = \sqrt{\sigma^2_x + 3\,\tau^2} \qquad (59)$$

[21] "Howe Bridge Trusses," by D. I. Jouravsky, Rept. of Petersburg–Moscow Railroad, Pt. II, Supplement III, St. Petersburg, Russia, 1855, p. 130.

[22] "Strength of Materials," by S. Timoshenko, D. Van Nostrand Co., New York, N. Y., Pt. II, 1941, p. 479.

in which, as will be noted, the yield of the material is dependent on both the normal and the shearing stress.

It appears that at present no complete solution for the problem involving shear is available. However, some studies indicate[23] that, even with shear, the assumption of plane sections remains essentially valid; but shearing stresses contribute to distortion. An attempt to derive a complete solution based on elastico-plastic considerations has been made by Anton Eichinger,[23] based on some earlier work at the Swiss Federal Testing Laboratories.[24] His solution is not in an applicable form to a practical problem, yet it clearly indicates the interdependence of normal and shearing stresses in the plastic range. Thus, the analyst cannot just "rotate" the tension test diagram to obtain stress distribution in the plastic range where shear occurs. The problem involves several variables and hence it is much more complicated than the one indicated by Professor Hrennikoff.

BERNARD A. VALLERGA,[25] JUN. ASCE.—In the "Introduction" Professor Hrennikoff states that "* * * a simpler case of the same problem, involving a rectangular beam, has been treated, in a somewhat different way, by S. Timoshenko.[3]" An analysis of Professor Timoshenko's method of reduced modulus will show that the two methods are identical in so far as the determination of the resisting moment of any section is concerned. Application of the additional simplifying limitation No. 2, as listed by the author, to Professor Timoshenko's more general expression will result in identical equations for bending moment. Hereinafter is a demonstration of the preceding statement for the two cases considered.

Rectangular Section.—Professor Timoshenko shows that, for a material not following Hooke's law, the resisting moment is given by the general expression,

$$M = \frac{E_R I}{r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

in which E_R is the reduced modulus and r is the radius of curvature of the beam due to the moment M. For a rectangular section,

$$E_R = \frac{12}{\Delta^3}\int_{\epsilon_2}^{\epsilon_1} \sigma\, \epsilon\, d\epsilon \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

in which Δ is the sum of the absolute values of ϵ_1 and ϵ_2. For the case of a mild steel whose compressive stress-strain diagram is assumed to be the same as the tensile stress-strain diagram (limitation No. 2),

$$\epsilon_1 = \epsilon_2 = \epsilon_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (62a)$$

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[23] "Biegung mit Querkraft, ausserhalb des Gebietes der rein elastischen Formandërung," by Anton Eichinger, 2d Cong., International Assn. for Bridge and Structural Eng., *Final Report*, Berlin, 1939, English Ed., pp. 43–47.

[24] Discussion by M. Ros and A. Eichinger of "La Stabilité de L'âme et des Ailes des Barres Comprimées," by F. Friederich Bleich, 1st Cong., International Assn. for Bridge and Structural Eng., *Final Report*, Paris, 1932, pp. 144–145.

[25] Lecturer in Civ. Eng., Univ. of California, Berkeley, Calif.

[3] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, p. 45.

and

$$\Delta = 2\,\epsilon_0 \ldots\ldots (62b)$$

Therefore,

$$E_R = \frac{3}{\epsilon^3_0}\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon \ldots\ldots (63)$$

However,

$$\frac{1}{r} = \frac{2\,\epsilon_0}{h} \ldots\ldots (64)$$

and

$$I = \frac{b\,h^3}{12} \ldots\ldots (65)$$

Substitution of Eqs. 63, 64, and 65 in Eq. 60 gives

$$M = \tfrac{1}{2}\,b\,h^2\,\frac{1}{\epsilon^2_0}\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon = \tfrac{1}{2}\,m_1\,b\,h^2 \ldots\ldots (66)$$

I-*Section.*—Professor Timoshenko's method can be expanded to include I-beams. Thus,

$$M = \frac{E_R\,I_w}{r} \ldots\ldots (67)$$

in which, for an I-section the general expression for reduced modulus is

$$E_R = \frac{12}{\Delta^3}\left(\int_{e_2}^{\epsilon_1} \sigma\,\epsilon\,de + \frac{b'-b}{b}\int_{e'_1}^{\epsilon_1} \sigma\,\epsilon\,d\epsilon + \frac{b'-b}{b}\int_{e'_2}^{\epsilon_2} \sigma\,\epsilon\,d\epsilon\right) \ldots (68)$$

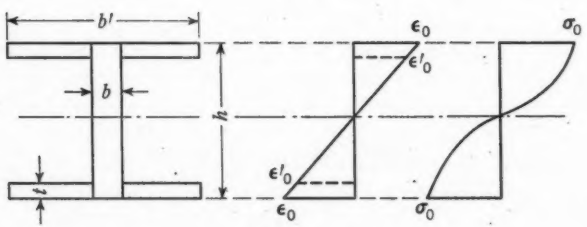

FIG. 38

If limitation No. 2 is applied to this case, together with the additional approximation that the stress does not vary across the flange (see Fig. 38), Eq. 68 becomes

$$E_R = \frac{3}{\epsilon^3_0}\left[\int_0^{\epsilon_0} \sigma.\,\epsilon\,d\epsilon + \frac{b'-b}{b}\,\sigma_0\,\frac{\epsilon^2_0 - (\epsilon'_0)^2}{2}\right] \ldots\ldots (69)$$

From the geometry in Fig. 1 it is evident that

$$\epsilon_0 - \epsilon'_0 = \frac{2\,t}{h}\,\epsilon_0 \ldots\ldots (70)$$

Substituting Eq. 70 in Eq. 9 after factoring the difference of the squares gives:

$$E_R = \frac{3}{\epsilon^3_0}\left[\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon + \frac{(b' - b)t}{\frac{1}{2}\,b\,h}\,\sigma_0\,\epsilon_0\,\frac{\epsilon_0 + \epsilon'_0}{2}\right] \ldots\ldots\ldots (71)$$

Introducing into Eq. 71 the factor κ as defined by Professor Hrennikoff, and making the further approximation,

$$\epsilon_0 + \epsilon'_0 = 2\,\epsilon_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (72)$$

then

$$E_R = \frac{3}{\epsilon^3_0}\left(\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon + \kappa\,\sigma_0\,\epsilon^2_0\right) \ldots\ldots\ldots\ldots\ldots (73)$$

Finally, by substituting Eqs. 64, 65, and 73 in Eq. 67, the bending moment becomes

$$M = \tfrac{1}{2}\,b\,h^2\left(\frac{1}{\epsilon^2_0}\int_0^{\epsilon_0} \sigma\,\epsilon\,d\epsilon + \kappa\,\sigma_0\right) = \tfrac{1}{2}\,m\,A_w\,h \ldots\ldots\ldots\ldots (74)$$

which is the same as Eq. 5.

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DISCUSSIONS

A RATIONAL EXPLANATION OF COLUMN BEHAVIOR

Discussion

BY FREDERICK L. RYDER

FREDERICK L. RYDER,[28] ESQ.—It is stated in the discussions, particularly by Professors Winter and Mains, that a good part of the material in the paper is to be found elsewhere in the literature. In view of this criticism, the writer wishes to clarify the purpose and scope of the paper, as follows:

It is well known that various aspects of the analytical treatment of the column problem have been investigated previously. Although these treatments have no doubt been adequate for those scholars who were able to integrate them into a comprehensive theory, the writer felt that in no case did they present to the average engineer a unified, cohesive theory of column behavior. As taught in the schools, for example, column behavior is discussed largely in terms of empirical results, presumably because there is no paper that presents a reasonably simple but well-rounded analytical treatment. The result is that most engineers do not understand the rationale of column mechanics.

To remedy this situation, the writer has attempted to develop, from the beginning, a logical and sequential column theory, geared to the needs of the modern engineer. Because of the analytic and self-developing nature of the work, it was unnecessary to use the methods and results of previous investigators, and in fact they were not used.

Aside from what he hopes is a more meaningful and useful organization of certain known results, the writer believes that certain parts of the paper are essentially new—for instance, the discussion of column stability in Section 3, which is essential to an understanding of column mechanics. Also, the essence of the step-by-step method appears to be entirely new, at least as applied to columns. In this connection, the writer wishes to point out a basic feature—namely, that his method is not one of successive approximations, and consequently does not require a good guess as a first approximation. This procedure

NOTE.—This paper by Frederick L. Ryder was published in March, 1947, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1947, by R. Robinson Rowe, and George Winter; June, 1947, by Robert M. Mains, and Chi-Teh Wang; and October, 1947, by Ernest L. Robinson.

[28] Development Engr., Shirgun Corp., New York, N. Y.

is in contrast with previous methods, such as that of N. M. Newmark,[29] M. ASCE. The difference would seem to be important in the case of complicated columns, as, for example, a cargo boom, which often consists of pipes of different sizes joined at their ends, with multiple hoisting blocks applying both lateral and eccentric axial loads at different points near the end of the boom.

The writer had also hoped that Fig. 3, which presents a convenient solution of slender columns loaded above the critical point, was new. However, it turns out that curves of a similar nature have been published by Mr. Robinson. The writer extends his apologies.

With regard to specific ideas presented in certain discussions, the writer wishes to reply as follows:

To Mr. Rowe he wishes to extend his thanks for the interesting historical notes.

Professor Winter states that a discussion of columns loaded above the critical point is of no practical value. The writer believes that a study of such columns is necessary to obtain a proper insight into the behavior of the more usual columns. Furthermore, such columns are often used as machine and instrument parts. This use, to be sure, is of interest to mechanical engineers rather than to civil engineers.

As Professor Mains declares, the writer did not extend his analysis to include the determination of the buckling load for columns of other than the ideal shape. The reason is simply mathematical complexity, for it would have led to the solution of Eq. 3 with the parameter I a function of x, and not necessarily continuous. Quite often columns of nonideal shape have nonideal loading, in which case they may be completely analyzed, both as to stress and as to deflection, by the step-by-step method.

Professor Wang indicates that, at values of P/A close to the critical, the deflection of an ideal column is a determinate problem, contrary to the writer's statement in Section 3, conclusion (3), and may be evaluated from Eq. 13. The writer believes nevertheless that his original statement is correct. A numerical investigation of Eq. 13 in the vicinity of $q = 0$ will show that, as q is varied, K varies by a lesser order of magnitude. This demonstration constitutes the mathematical and physical definition of indeterminateness of q, and hence of deflection.

Professor Wang's clarification of the behavior of the column between the proportional limit and the yield point is most enlightening from a rational point of view, and would seem to be of definite value in close design.

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[29] "Numerical Procedure for Computing Deflections, Moments, and Buckling Loads," by N. M. Newmark, *Transactions*, ASCE, Vol. 108, 1943, p. 1161.